The width of the arches

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dedicated to Ron Lancaster

In September 06 I was at the King Fahd University in Saudi Arabia for a workshop on preparing high school students for university. Though Saudi Arabian culture is very different from ours, they have the same problems in getting their students excited about the study of mathematics, and indeed that was the focus of the workshop. The picture below shows me standing inside a particularly fine walkway together with (from left to right) Dr. Hussain Al-Attas, the Director of First-year Studies and Dr. Suliman Al–Homidan, the Head of the Mathematics Department. I had my colleague Peter Galbraith (from Australia) take the picture thinking that there must be a good math problem somewhere.

And indeed there is! recently I gave it to a mixed group of grade 11 and 12 students and what happened then was quite fascinating. Here's the problem.

Problem
The arches that recede into the background of the photo are equally spaced and are the same size. But their apparent size on the picture decreases. The question is, exactly how do they decrease?

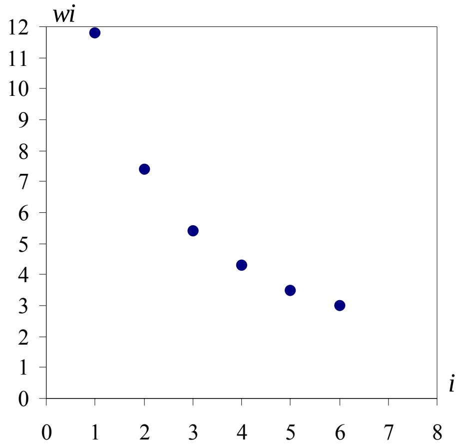
(a) Take the measure of "size" to be the width of the horizontal opening of the arch as measured on the page by a ruler. Indeed let w_i be the width of the *i*th arch on the page (use the front face of the arch). I want you to give me the form of this dependence—that is, how does w_i depend on i. Your expression for w_i will have some parameters as there are physical measurements you are not given. So what I really want here is the *functional form* of the dependence.

(b) Suppose I tell you that the arches are spaced at intervals of 2.5 meters. How far away is the camera from the front face of the first arch?

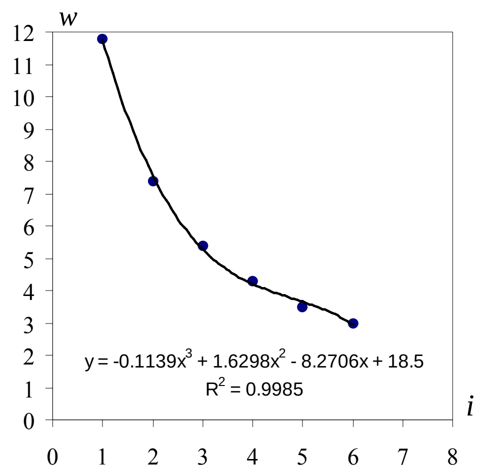
Discussion and solution
(a) The students' failures are perhaps even more interesting than their successes and I will begin with these. Most of them hardly knew where to begin. There was nothing here they recognized as a math problem with a known method of solution.

Finding the best curve.
The idea of actually gathering some data (making measurements) caught on, and soon everyone had a list of widths for at least 6 arches. [Note that the students were working with a full page photo, slightly bigger than the one on the first page.] I had an excel sheet running on the data projector, and the data was plotted for all to see. What happened next was (to me) completely unexpected. The students started vying to see who could produce the function that gave the best fit. Well that shouldn't have surprised me at all. They are after all exactly what we have made them to be—true children of technology. So let me begin with that story.

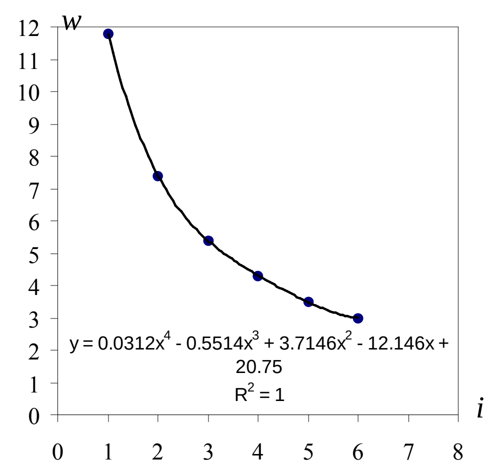
i	w_i
1	11.8
2	7.4
3	5.4
4	4.3
5	3.5
6	3.0



What kind of curve does that data set evoke? Believe it or not, some students tried polynomials. A cubic polynomial gives an impressive $R^2 = 0.9985$, but a quartic gives $R^2 = 1$, and you can't do any better than that. The student in question deduced that the right answer must be the quartic polynomial shown below at the right. [One thing that has to be emphasized at some point, though perhaps not now, is that this is definitely not what R^2 is supposed to be about.]

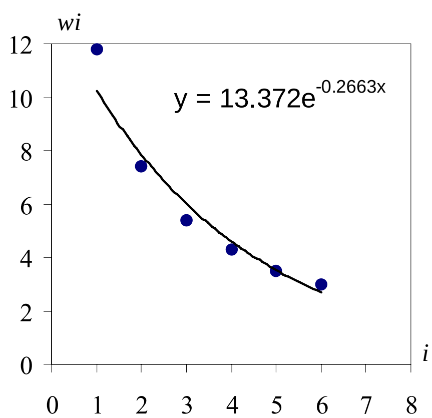


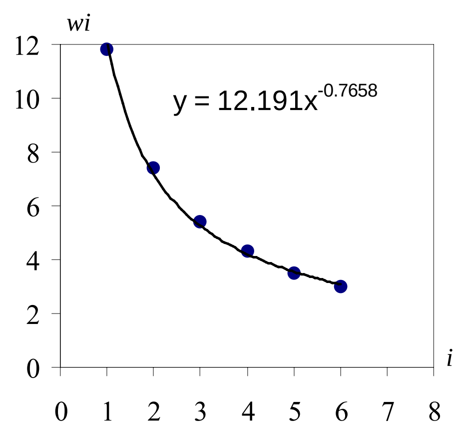
A cubic fit



A quartic fit

Two other popular choices were the exponential and the power function seen below left and right. The exponential form, $w_i = ar^i$, is clearly a bad fit, but some of its proponents remained stubbornly committed to it none-the-less. One group calculated a number of the candidate ratios $r = w_1/w_2$, w_2/w_3, w_3/w_4, etc. and actually concluded from this that the ratios were *not* constant. I gave them thumbs up for that one! The power function looks pretty good and it is fact close in spirit to the correct answer.

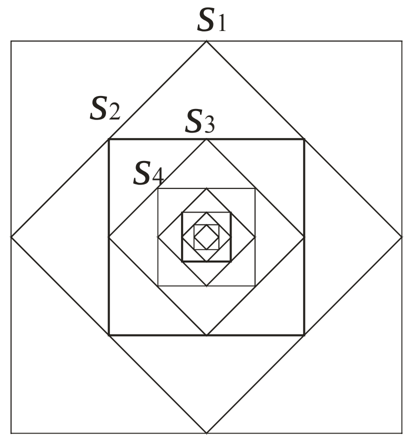




At this point, the class took a vote and chose the power function. It doesn't give the perfect fit of the polynomial, but it's simpler and I guess they thought it had a better chance of being right (or being the answer I wanted them to find?).

What was I to do? I had a class of curve-fitters on my hands. How was I to get them to imagine that there was a totally different way to tackle the problem?

What I did was to put the following problem on the board. Suppose I gave you the following nested sequence of squares and asked you to find me a formula for the side-length s_i of the *i*th square. Suppose the size of the largest square is s_i = 4 cm. What's the size of the *i*th square?

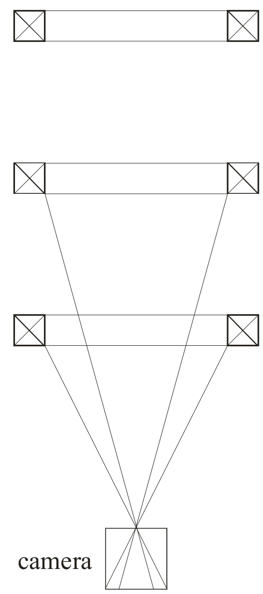


What would you do? Would you measure the sequence of side lengths, and then try to find a curve which gave a good fit?

They stared at the board for a while and slowly they got the point. No. They'd start to do some geometry. Pythagoras. Stuff like that.

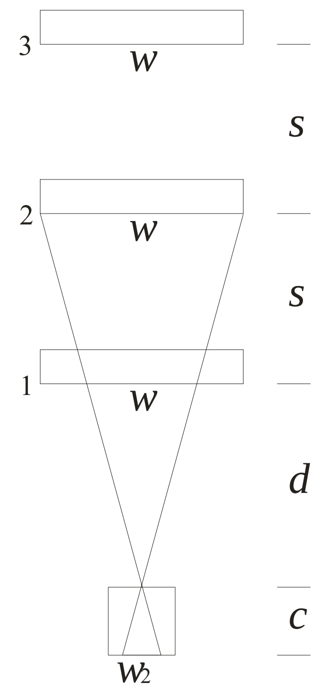
Okay, I said. Is the arches problem maybe something like that? How was the photo made?

At last we started a conversation about cameras and light rays.



The geometers
My own thoughts about the problem concluded that the diagram at the right is the one that needs to be drawn. What is significant about it is that it includes the camera and its components, like the aperture and the film. Some students represented the camera as a point and they were right away prevented from thinking clearly about *what was really happening*, and so it was hard to make any sense of their "arguments."

When the diagram is made into a schematic and variables introduced, then similar triangles immediately give us the form of the equation relating w_i to the width w of the arches. For example, the second arch gives us:



$$\frac{w_2}{c} = \frac{w}{d+s}$$

Here c is the depth of the camera, d is the distance of the camera from the first arch and s is the spacing between arches. The corresponding formula for the ith arch ($i \geq 1$) is

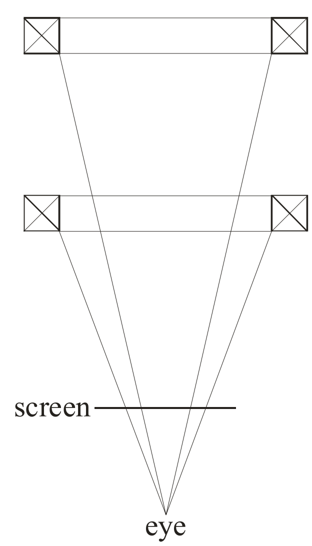
$$\frac{w_i}{c} = \frac{w}{d+(i-1)s}$$

$$w_i = \frac{cw}{d-s+is}\,.$$

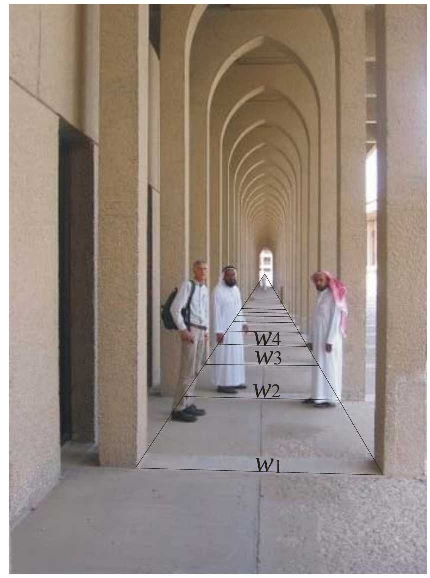
This provides the form of the dependence of w_i on i. More abstractly, we could say that it has the form:

$$w_i = \frac{1}{a+bi}\,.$$

A few students worked with a diagram like that at the right, thinking of the image as being captured on a "screen" placed in front of the eye. There's no camera here but they are thinking of the image as if the viewer looked at the scene through a window with the image etched on a pane of glass. That certainly works for me and this solution got "full marks."

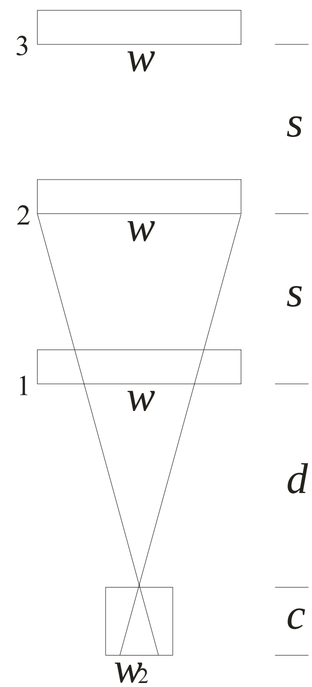


A number of students worked with the diagram they simply drew on the picture. There's lots of similar triangles here, and lots of chance to work with angles and use results from trigonometry, and it was a bit heartbreaking to see them struggle away, saying things about perspective and the significance of the point at infinity, and writing down everything they thought might be relevant but without any hope of getting a coherent solution.



What do I conclude from this?—that from an early age our students need more experience in doing what mathematicians do best, shining a careful precise mathematical light into a corner of the world and coming up with an explanation of what's really happening, and perhaps why.

(b) Here we are told that the arches are spaced at intervals of 2.5 meters, and are asked for the distance from the camera to the front face of the first arch. We are given $s = 250$ (cm) and we are asked to find d, but cw is unknown. However taking the quotient of two successive w_i will eliminate it. Most students did this and chose w_1 and w_2:



$$\frac{w_1}{w_2} = \frac{d+s}{d}$$

$$dw_1 = (d+s)w_2$$

$$d = \frac{sw_2}{w_1 - w_2}$$

Using the values from part (a), working in cm:

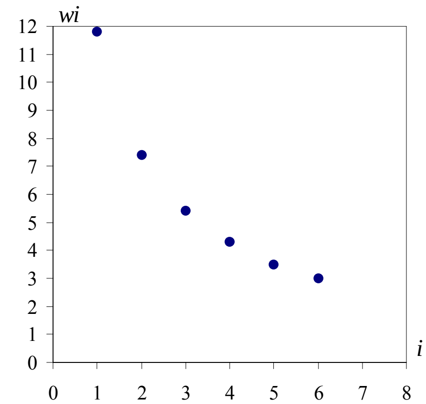
$$d = \frac{250 \times 7.4}{11.8 - 7.4} = 420$$

The camera is 4.2 meters from the first arch.

By the way, those who used the "screen" diagram got the right answer if they calculated the distance from the first arch to the eye.

Of course I gave this solution full marks. But I am myself partial to an approach which uses *all* the measurements that were made. And as a bonus it can give us a verification of our model.

Finding a straight line.
Whenever I'm fitting a curve to data I like to transform the data if possible so that the transformed graph ought to be a straight line. The reason for that is that straight lines are easy to "see" and can therefore provide a simple check on the model. In addition, the equation of the trend-line, in using all the data, can give us "best-fit" values of our parameters.

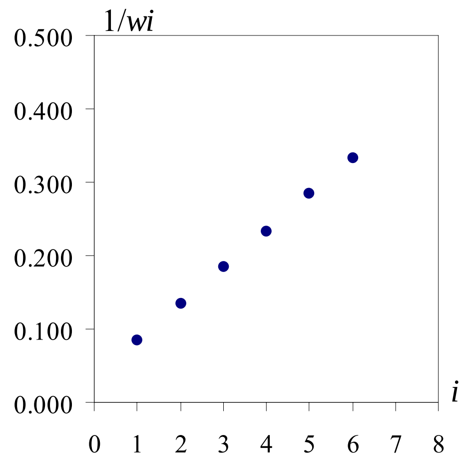


In this case we have a functional relation between w_i and i of the form

$$w_i = \frac{cw}{d - s + is}.$$

Can we get a form which is linear in i? Certainly—by working with $1/w_i$:

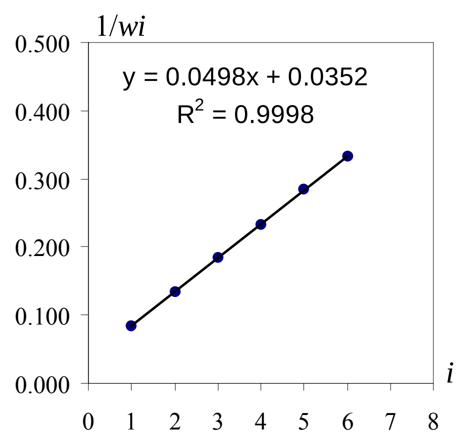
$$\frac{1}{w_i} = \frac{d-s}{cw} + \frac{s}{cw} i$$



This tells us that a plot of $1/w_i$ against i should be a straight line. And lo and behold it is so. *Very pretty.* That's a very satisfying way to validate our geometric argument.

One thing we might do now is get best-fit values for our parameters from a trend line. Technology gives us the line:

$$\frac{1}{w_i} = 0.0352 + 0.0498i$$



Comparing equations:

$$\frac{d-s}{s} = \frac{352}{498}$$

$$498d = (498+352)s = 850s = 850 \times 250$$

$$d = 850 \times 250/498 = 427.$$

The answer we got before, using only w_1 and w_2, was $d = 420$. The difference between the two approaches is only 7 cm, but which one is apt to be better? That's an interesting question…

Which of the two answers is better?—the first one that used only w_1 and w_2, or the second one that used all the data? One might argue for the second as it uses more data, however "best-fit lines" (which are now available at the touch of a student's fingertips) need to be used with care. One key assumption behind linear regression is that the size of the measurement error is the same for all data points and this is not likely true for the values of $1/w_i$. If the w_i were all measured with the same accuracy, the error in $1/w_i$ would be much bigger for the arches that are farther away (can you see why this is the case?) and the theory says that in the "least-squares" minimization, these would need to be given less weight (and the standard regression line doesn't do that). So of our two solutions, the first one gives the first two arches too much weight and the second gives them too little weight. Maybe the real answer is somewhere in-between!