Boys and girls (conditional probability)

Example 1. *This example provides a standard technical introduction to the subject. What it may lack in fascination it makes up for in the direct visual interpretation of conditional probability.*
(a) I roll a pair of dice. Find the probability that their sum is 9.
(b) I roll a pair of dice and I tell you that the outcomes differ by 1. Given this information, find the probability that the sum is 9.

[There is some terminology here. Part (a) asks for the probability of rolling 9. Part (b) asks for the probability of rolling 9 conditional upon rolling a difference of 1.]

Solution.
(a) There are 36 possible pairs, all equally likely, corresponding to the entries in the table. Of these, 4 possibilities (marked with an ×) have sum 9, so the probability is 4/36 = 1/9.

	1	2	3	4	5	6
1						
2						
3						×
4					×	
5				×		
6			×			

(b) The given information has excluded some of the outcomes in the table; in fact only the 10 shaded squares can occur. Of these, 2 possibilities have sum 9, so the probability is 2/10 = 1/5.

	1	2	3	4	5	6
1						
2						
3						×
4					×	
5				×		
6			×			

It's important to understand what happened here. We calculated (a) as the *proportion* of the 36 (equally likely) possibilities that are successful. We calculated (b) also as a proportion, except the effect of the information was to restrict our attention to a *subset* of the possibilities.

That's how conditional probability works—some information is given in terms of a condition that must hold and we identify the subset of the possibilities that satisfies the condition, and then solve the problem using that as our "universe."

Example 2.
This is a simple problem, but "the man on the street" often gets it wrong. What you have to do is think clearly about what everything means.
For this problem assume (though it's not quite true) that any child is male or female, independently of any other, with probability ½.
(a) You know only that your colleague has three children. What is the probability that they are all girls?
(b) You know only that your colleague has three children. You ask him if the oldest one is a girl and he says yes. Given this information, what is the probability that all three children are girls?
(c) You know only that your colleague has three children. You ask him if he is lucky enough to have at least one daughter and he says yes. What now is the probability that all three children are girls?

(a) There are two ways to reason. First, since the chance of a child being a girl is 1/2, the chance of all three being girls is (1/2) (1/2) (1/2) = 1/8. Alternatively, make a list of all possible outcomes, in order of age: {GGG,GGB,GBG,BGG,GBB,BGB,BBG,BBB}. As each one is equally likely, the chance of GGG is again 1/8.

(b) Again, we can reason in two ways. If we know the oldest one is a girl, then the chance that all three were girls is simply the chance that the last two children were girls, and this is (1/2) (1/2) = 1/4. We can also use the list of cases in (1a). Since the first born is a girl, the only cases of consideration are GGG, GGB, GBG and GBB, and only 1 of these 4 is GGG. Thus the chance is simply 1 in 4.

(c) Many people argue that the probability here is again 1/4. But it's not. The information that "there is at least one girl" only removes the outcome BBB, leaving us with the remaining 7, all equally likely. As only one of these is GGG, the probability is 1/7.

Example 3. *The problem with Mary*. Here's an intriguing problem I found on the web. Again assume that each child has an equal chance of being male and female.
(a) A mother has two children. The older one is a daughter. What is the probability that the other child is a girl?
(b) A mother has two children. The older one is a daughter named Mary. What is the probability that the other child is a girl?
(c) A mother has two children. One of them is a daughter. What is the probability that the other child is a girl?
(d) A mother has two children. One of them is a daughter named Mary. What is the probability that the other child is a girl?

Solution.
(a) The answer is 1/2. The sex of the older has no effect of the probabilities for the younger.

(b) The answer is 1/2, for exactly the same reason as (a).

(c) The possibilities for 2-child families are BB, BG, GB and GG all equally likely, where I have listed the eldest first. The information given restricts the sample space to the last three: BG, GB and GG, still all equally likely. Of these only one GG has the required property. The answer is 1/3.

(d) The first thing to say is that the situation is not tightly specified. You need to make an assumption about the naming of daughters. For example, suppose that the mother of Jesus was the second-born girl of her family and therefore Christian mothers had a tendency to name their second-born daughter Mary. That would certainly affect the answer. So we assume that mary is a randomly chosen name. To make this a bit tighter still, assume that there is a pool of (say) 1000 female names and every time a daughter is born the mother chooses a name at random from the pool. To be even tighter let's also assume that a GG mother would not name *both* of her daughters Mary, so that if Mary was the name she chose for her first child and then she drew for her second daughter and got (with probability 1/1000) Mary again, she'd reject it and keep drawing until she got another name. Okay, now we have a well-defined question. What's the answer?

Well let's tabulate the different kinds of 2-child families with a Mary, and write the probability of each. They are (with eldest listed first),

MB BM MG GM

The first is a GB family who chose Mary as a name and the probability of that is

$$(1/4)\times(1/1000) = 1/4000.$$

The second is a BG family who chose Mary as a name and the probability of that is the same:

$$(1/4)\times(1/1000) = 1/4000.$$

The third is a GG family who chose Mary as the first name and the probability of that is the same:

$$(1/4)\times(1/1000) = 1/4000.$$

The fourth is a GG family who chose Mary as the second name and the probability of that turns out to be the same again though maybe that's not quite so obvious. If you like you can "take it apart." We start with a GG family with probability ¼. Then the first name was a non-Mary and the probability of that is 999/1000. The second name is now chosen randomly from the pool that now contains only 999 names (the first child being omitted) and Mary will get chosen with probability 1/999. The overall probability is:

$$(1/4)\times(999/1000)\times(1/999) = 1/4000.$$

Okay. The four cases are all equally likely. In the last two the other child is a girl. So the probability is 2/4 which is 1/2. *It is surprising, that (a) and (b) have the same answer, but (c) and (d) do not.*

Problems

1.(a) I roll a pair of dice. What is the probability that the outcomes differ by exactly 2?
(b) I roll a pair of dice and I tell you that the sum of the outcomes is at least 6. What now is the probability that the outcomes differ by exactly 2?

2. You know that your colleague has two children and you also know that he has a daughter. You are wondering whether or not both his children are girls, and based on the evidence so far, you assess the probability of that to be ½. The other day you spy him dropping one of his children off at school and you see that it is a girl. You have a feeling that this observation should increase the probability that both his children are girls. Does it? If so, what might the new probability be?

3. *The fifth of Lewis Carroll's Pillow Problems*. A bag contains a counter, known to be either white or black with probability 1/2 each. I put a white counter in, and then draw a random counter out of the bag, which proves to be white. What now is the chance that the remaining counter is white? [Of course, the bag is now identical to what it was before (white counter in; white counter out) so the probability that the remaining counter is white is 1/2, the same as it was before. Right?]

4. *A classic card puzzle*. I have three cards, one with both sides red, one with both sides blue, and one with one red side and one blue side. I hold up a random card with a random side facing you. If the side you see is red, what is the probability that the other side of the same card is red?

5. I deal two cards from the top of a well shuffled deck.
(a) What is the probability that they are both aces?
(b) I tell you that the first card dealt is a spade. What now is the probability that they are both aces?
(c) I tell you that the first card dealt is a spade and the second card dealt is a heart. What now is the probability that they are both aces?

6. Now for an interesting sequence of problems. You'll need some counting skills from **Pascal**, essentially that the number of ways of choosing r objects from a selection of n different objects is $\binom{n}{r}$, the rth entry in the nth row of Pascal's triangle. For example, $\binom{5}{3} = 10$, and in general: $\binom{n}{r} = \frac{n(n-1)\ldots(n-r+1)}{r(r-1)\ldots1}$. In this problem we extract 16 cards from a standard deck, the 4 Aces, the 4 Kings, the 4 Queens and the 4 Jacks. Suppose I shuffle these and deal myself 4 cards at random.

<u>Q1</u>. You choose a random card from my hand and it is the ace of spades. What is the probability that the hand contains only one ace?

<u>Q2</u>. You choose a random card from my hand and it is an ace. What is the probability that the hand contains only one ace?

<u>Q3</u>. You ask me if the hand contains the ace of spades and I say yes. What is the probability that the hand contains only one ace?

<u>Q4</u>. You ask me if the hand contains at least one ace and I say yes. What is the probability that the hand contains only one ace?