Bridge

Example 1. Find the probability that a bridge hand contains no aces.

Solution. My students all have an intuitive grasp of what probability means but have trouble translating that into computational action. The point is that a probability is a proportion—the probability P of getting a no-ace hand is the proportion of all possible hands that have no aces. Now in **Pascal** we have calculated the total number of bridge hands as $\binom{52}{13}$.

Now we ask how many of these contain no aces.

Well, to make such a hand (with no aces), I've got to choose 13 cards from 48 (52 – 4 aces = 48) so the number of such hands is $\binom{48}{13}$. So the *proportion* of all hands that have no aces is

$$P = \frac{\binom{48}{13}}{\binom{52}{13}} = \frac{48 \times 47 \times \ldots \times 36 / 13!}{52 \times 51 \times \cdot \ldots \times 40 / 13!} = \frac{39 \times 38 \times 37 \times 36}{52 \times 51 \times 50 \times 49} \approx 0.30$$

There's a 30% chance of getting no aces. Notice the cancellation that took place in the quotient. Even if you have an $\binom{n}{r}$ button on your calculator, it's usually a good idea to do the cancellation before you calculate, especially when it gives a simple expression for the answer. Such expressions are often revealing, and can help in troubleshooting for errors.

Example 2. Find the probability that a bridge hand contains the ace of spades.

Solution. To get a hand containing the ace of spades I need to choose 12 cards from the remaining 51, giving $\binom{51}{12}$ possible hands. The probability is

$$P = \frac{\binom{51}{12}}{\binom{52}{13}} = \frac{51 \times 50 \times \ldots \times 40 / 12!}{52 \times 51 \times \ldots \times 40 / 13!} = \frac{13}{52} = 0.25$$

Probability and Counting

A number of nice probability problems come down to counting. We want to calculate the probability of a "success" and we count the number of successful configurations and the total number of configurations and take the ratio.

$$P = \frac{\#\,\text{successes}}{\text{total}\,\#}$$

What we've essentially done is to model a probability as a <u>proportion</u>. In **Pascal** we learned how to count. We now make use of this to solve some probability problems.

Behind this "proportionality" method is the hidden assumption that all the configurations that we are counting have the same chance of occurring. In this case, we need to know that all hands have an equal chance of occurring.

When this assumption fails we have to "weight" each configuration by its probability of occurring (e.g. see **hitting 10**)

Exactly a quarter! Well that's curious—there must be something simple going on here. And indeed there is—a hand is just a quarter of the deck, and so it should contain any fixed card with probability 1/4.

Example 3. Find the probability that a bridge hand contains all four aces.

Solution. To make a hand containing all 4 aces, I have to choose 9 more cards from a pool of 48 possibilities, so the number of successful hands is $\binom{48}{9}$. The proportion of successful hands is:

$$P = \frac{\binom{48}{9}}{\binom{52}{13}} = \frac{48 \times 47 \times \ldots \times 40 / 9!}{52 \times 51 \times \ldots \times 40 / 13!} = \frac{13 \times 12 \times 11 \times 10}{52 \times 51 \times 50 \times 49} \approx 0.0026$$

That's about 1/4% or roughly 1 in every 400 hands.

Example 4. Find the probability that a bridge hand contains exactly two aces.

Solution. Start with a fixed pair of aces, say the heart and the spade. To complete the hand we need 11 more cards chosen from 48, so there are $\binom{48}{11}$ hands that contain these two aces. It will be the same for any pair of aces. Now how many such pairs are there?—the number of ways of choosing 2 objects from 4, which is $\binom{4}{2}$=6. Hence there are $\binom{4}{2}\binom{48}{11}$ hands that contain exactly 2 aces. The *proportion* of hands containing exactly 2 aces is then:

A new idea is needed for this problem, and it's this: there are 6 possibilities for what those two aces might be—calculate the probability for each such pair and multiply by 6.

$$P = \frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}} = 6\frac{48 \times 47 \times \ldots \times 38 / 11!}{52 \times 51 \times \ldots \times 40 / 13!} = 6\frac{39 \times 38 \times 13 \times 12}{52 \times 51 \times 50 \times 49} = 0.21\,.$$

The chances of getting exactly two aces are about 20% or roughly 1 in every 5 hands.

Example 5. Find the probability that a bridge hand contains exactly one ace.

Solution. Modifying Example 4. the answer is

$$P = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} = 4\frac{48 \times 47 \times \ldots \times 37 / 12!}{52 \times 51 \times \ldots \times 40 / 13!} = 4\frac{39 \times 38 \times 37 \times 13}{52 \times 51 \times 50 \times 49} \approx 0.44$$

There is a 44% chance of getting exactly one ace.

This is interesting. A bridge hand contains a quarter of the deck, and a quarter of all the cards are aces, so shouldn't every hand contain an ace? No but what *is* true is that every hand contains an ace <u>on average</u>, or more precisely the average number of aces per hand is 1. If all hands had the same number of aces, then they would indeed all have exactly one ace, but since some hands have more than one, some will have none.

Example 6. Find the probability that a bridge hand contains at least one ace.

Solution. Something interesting often happens with this problem. As I wander around the room, I find several groups producing the answer $4\binom{51}{12}$. I ask someone to present the argument and it goes like this. You need an ace, so take one out, and put it aside—there are four possibilities for that. Then to make up the rest of the hand, you need to choose 12 cards from the remaining 51.

Is this argument correct? That's a good question. Even those students who feel there's something wrong with it have a hard time getting hold of the flaw.

Is this a sound argument? How do we troubleshoot it? Well here's a nice approach. What the argument is saying is that there are $\binom{51}{12}$ hands that contain the ace of spades, and the same number that contain the ace of hearts, and the same for clubs and diamonds (okay so far?—yep!), and to get the number of hands that contain at least one ace, you add those four (identical) numbers up. Right?

Wrong. When we say it out like this, we can see that any hand that contains more than one ace will get counted more than once, so we have over-counted the hands that contain at least one ace. So our argument is flawed. It's possible to patch it up by subtracting off the hands that get counted two or three or four times, but there's a much easier approach.

Simply observe that a hand either contains at least one ace, or it contains no ace. Now we counted the latter kind in Example 1—there are $\binom{48}{13}$ hands with no ace. So there must be $\binom{52}{13}-\binom{48}{13}$ hands with at least one. Thus the proportion that contain at least one ace is

$$P = \frac{\binom{52}{13}-\binom{48}{13}}{\binom{52}{13}} = 1-\frac{\binom{48}{13}}{\binom{52}{13}} = 1-\frac{48\times 47\times \ldots \times 36/13!}{52\times 51\times \ldots \times 40/13!}$$

$$= 1-\frac{39\times 38\times 37\times 36}{52\times 51\times 50\times 49} \approx 1-0.30 = 0.70$$

You can see from the structure of the argument that it can be made directly in terms of probabilities, and so the answer can be obtained directly from Problem A:

P(at least 1 ace) = 1 – P(no ace) ≈ 1 – 0.30 = 0.70.

Any probability problem that contains the phrase "at least one" can generally be done this way.

Problems

1.(a) How many cards must I deal for the probability of getting at least one spade to exceed 1/2?

(b) How many cards must I deal for the probability of getting at least one ace to exceed 1/2?

2. What is the probability that a bridge hand will contain

(a) no black queen?
(b) at least one black queen?
(c) the queen of spades?
(d) both black queens?

3. What is the probability that a bridge hand will contain

(a) only black cards?
(b) at least one black card?
(c) more black cards than red cards? (a little thought pays dividends)

4. What is the probability that a bridge hand has a 3-3-3-4 distribution (i.e. at least 3 of each suit).

5. Crazy bridge is played using only the tens, the aces, and the face cards—thus there are 20 cards and 4 players, giving 5 cards per hand. Provide solutions to all the Examples 1-6 with "bridge" replaced by "crazy bridge."

6.(a) I deal from the top of the deck until I get to the ace of spades. What is the probability that the *next* card is also an ace? [Treat the deck as a cycle, so that if the spade ace happens to be the last card, the "next" card is the top card.]

(b) I deal from the top of the deck until I get to an ace. What is the probability that the *next* card is also an ace?

(c) I deal pairs of cards from the top of the deck until I put down a pair that contains the ace of spades. What is the probability that *both* cards of the pair are aces?

(d) I deal pairs of cards from the top of the deck until I put down a pair that contains at least one ace. What is the probability that *both* cards of the pair are aces?

This is a delicious sequence of problems, with a definite conditional-probability flavour to it. Actually it's an old chestnut, but I have roasted it just a bit.