Card Trick

This card trick is found in Ball and Coxeter Mathematical Recreations & Essays. *[University of Toronto Press, 12th ed. 1974. page 18]. By the way, don't let the title of this classic paperback (first edition 1892) fool you—it's got enough in it to fuel an undergraduate curriculum*

I give a deck of cards to a student and instruct her to perform the following sequence of operations. First she shuffles and deals 26 cards *face-up* in a stack. As she deals I watch the cards go down with great concentration. Then I ask her to turn the stack over—it is now face down and will be called the discard stack. She now deals the remaining deck, from the top, face up, into a number of piles, as follows. Suppose the first card dealt is a 7. Then she says "7" and covers it with 3 cards (face up), counting out "8," "9", and "10," as she deals. Then she starts a new pile. If the next card is a 4, she says "4", and covers it with 6 cards, calling out, "5, 6, 7, 8, 9, 10." If she lays down a 10, she just announces "10" and leaves it in a pile by itself. An ace counts 1, and all face cards count 10. At the end, she will usually run out of cards before she can finish a pile. In that case, the pile can't be used and she should pick it up and place it (face down) on the discard stack.

Now she chooses any 3 of the piles, and turns them over (so they are face down) and then turns over the top card on each of the three piles. She then gathers up all non-chosen piles and places them on top of the discard stack (face down). Now we add the values of the three turned-over cards and I concentrate for a moment and announce the name of the card that is in *that position* in the discard pile. For example, if the three turned-over cards were 4, 7, and Q, we would calculate 4+7+10=21 and then I would announce the identity of the 21st card (from the top) in the discard pile. The student then counts out 21 cards and encounters the named card. The effect on the class is quite striking.

I perform the trick a couple of times, and then I ask: how does it work? This is an ideal small group activity.

After some experimentation, the students will discover what is at first quite unexpected, that no matter what cards come up in the second stage, the named card is always the same card!—and that's the seventh card dealt at the very beginning. So to perform the trick, all you need to do is to notice and remember the 7th card dealt.

After they have made this discovery, the mathematician in me wants to make them turn to the question of "why does it work?" But in my experience, the students would much rather spend some time at this point performing the trick for one another and thinking about possible variations, and anyway that turns out to be valuable preparation for a general analysis.

Variations: The trick as I described it above has a couple of parameters which could be changed. Instead of counting to 10 to make the piles, why not count to $c = 13$? And instead of retaining 3 piles, why not take only $p = 2$? In fact you take other values for c and p as well (though there's only a small range that works well) and provided you know these at the beginning, you can still do the trick.

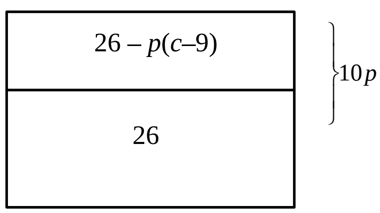
But with different c and p, the card to look for won't be the 7th anymore, but something else. Somewhere around there's presumably a "formula" for the number n of the card dealt in terms of c and p, and the inspiration to find that formula might well come from the class. That is, there should be some expression for n in terms of c and p:

$$n = f(c, p)$$

such that if we take c=10 and p=3, we get n=7, that is, $7 = f(10,3)$.

Given that we are prepared to accept the fact that *n* is independent of what cards are dealt in the second stage, we can find the formula by choosing a particularly "nice" outcome. For example, for the standard trick $(c = 10, p = 3)$, suppose that the three piles picked all happen to have top card 10. Then in fact they each have only 1 card, and the number of cards added to the discard stack is $26 - 3 = 23$. Now we want to count $3 \cdot 10 = 30$ cards off the top of this pile, and so the card we will get to will be $30 - 23 = 7$ deep in the original discard stack. So that's how the 7 can be obtained from the *c* and the *p*.

Now let's run carefully through that scenario for a general $c \geq 10$ and a general p, and see if we can obtain a formula for the depth *n* of the designated card. Suppose that the *p* piles picked all have top card "10", that is, a card worth 10 was dealt to start the pile). Then each of these piles will have $c - 9$ cards, and the number of cards added to the discard stack will be $26 - p(c - 9)$. Now we want to count 10*p* cards off the top of this pile, and the depth *n* of the card in the original discard stack will be given by the formula below.



$n = 10p - (26 - p(c-9))$
$= p(1+c) - 26.$

Check that if $p = 3$ and $c = 10$ we get $n = 7$, as before.

This is not an easy exercise; it requires a careful keeping track of things, and the students will find it quite demanding. But the more I think about it, the more I am convinced that the skills that are embodied here are precisely those that the are needed in virtually every occupation, and the importance of mathematics in the curriculum is not so much for the technical skills that are learned, as for the opportunity the subject provides to make an analysis of this type.

But now we have a general formula for the card to look for—valid for any given *p* and *c*.

For example, for $c = 13$ and $p = 2$, we get

$$n = 2(14) - 26 = 2$$

and we must look for and remember the second card dealt. Try it and see!

An interesting configuration with some creative possibilities is $c = 12$ and $p = 2$, since in this case

$$n = 2(13) - 26 = 0 \ .$$

A zero value for *n!* What can that mean?

Well, first of all, it means that I don't need to see anything in the discard stack at all—for example, I can watch it go down blindfolded, and claim I have powers to see through the blindfold. But what *do* I have to do?—I have to notice the identity of the card that is laid "first" on the discard stack (immediately on the top of the original 26-card stack). This is usually easy to do because the cards that are gathered are lying face up, and they will be usually be gathered face-up, and then turned over to be placed on the discard pile. Or if I want to play it safe, I can gather these piles up myself.

Now what of the task of constructing a proof that *n*=7 is independent of what cards appear when the piles are dealt? This is a suitable project for senior students, and there are a number of ways to go about it. One nice approach is inductive. Start with the above calculation which assumes that all three chosen cards were 10, and in this case the above formula for *n* applies giving *n*=7. Then look at the effect of changing one of those cards to a 9. This means we will count 1 less card off the top of the final discard pile, but it also puts one less card on top of the original discard stack, *and the value of n will remain unchanged*. Now any set of three top cards can be obtained from three 10's with a sequence of such steps. The same argument works in general for and *c* and *p*.

Why is this a good problem? The best reason is that the magic aspect makes it just about the most enticing problem I've ever used, certainly for a broad range of students. It's also very much "hands on," in that strategies and arguments are constructed with a deck in the hand. The construction of the general formula seems to use the skills of organization, abstraction and generalization that we hold to be important in mathematical development. For that matter, there's a lot of learning involved simply in imagining that looking for a general formula might be a reasonable thing to do.

And it's infectious. I did it one morning in a local high school, and saw it being performed to friends around the hallways at lunch

Problems

1. Provide the details of a trick, analogous to the one above, which would be done with a deck of 40 cards, obtained by simply removing the face cards, jack, queen and king. Count each pile to $c=10$, and use $p=2$ piles. What card should you look for in the initial deal?

2. Here's a cute little manipulation—let's see if you can figure out how it works. Take 16 cards. Divide it into two piles noting the number of cards in the smaller pile. As a specific example, suppose it's 3. Take up the bigger pile (with 13 cards) and hold it face up. Now deal out the cards face down, noting the identity of the 3rd card dealt (which will wind up 3rd from the bottom on the pile). Now take up the pile, still face down, and take the cards off the top one at a time, alternately putting them on the bottom of the deck and throwing them away. That is, the 1st card is put on the bottom, the 2nd is thrown away, the 3rd card is put on the bottom, the 4th is thrown away, etc. Then after a while there will be only one card left in your hand, and that will be the card you have noted. How does it work?

[Hint: One key condition is that 16 is a power of 2, and the trick can in fact be done with any other power of 2, e.g. 8 or 32. And that observation is the key to finding a nice recursive argument for why it works. In the final stage, when you are going through the deck, at one point you will have gone through the deck exactly once, so that the original top card is again on top. Show that at this point the configuration is exactly the same as it would have been at the start if you were using a deck of size 8. And then after another cycle, you have arrived at the beginning of a 4-deck trick. Etc.]

3. Here is one of the most incredible results I've seen in a long time. As a card trick it's spectacular; as a mathematical result, it's awesome. Now as a card trick, you can't actually perform it unless you can do a perfect shuffle, but you can still enjoy the mathematics.

So what's a perfect shuffle? Well, you divide the deck exactly in half, 26 in each hands, and you riff-shuffle so that the cards fall alternately. Actually, there are two kinds of perfect shuffles: in-shuffles and out-shuffles. With an out-shuffle, the top card stays on top, so that if the original deck is numbered from the top, 1 to 52, the shuffled deck will have the order: 1,27,2,28, etc. With an in-shuffle, the top card goes inside, so that the shuffled deck will have the order: 27,1,28,2 etc.

Okay. There are many instances in which a magician wants to perform a sequence of shuffles which will move the top card a certain distance down the deck, for example, suppose he wants to move it down 13 cards (from position 1 to position 14). The question is, can he do this with a sequence of perfect shuffles, and if so how? And the answer is, yes he can, but what's amazing is how you find the sequence of ins and outs that will do it. You first take the number 13 and write it in binary: 13 = 1101. Then you change every 1 to an I and every 0 to an O and perform the sequence IIOI of shuffles, where I, of course, means in-shuffle, and O means out-shuffle. That's it! As a small extra, it turns out that that also gives you the *least* number of shuffles that will do the job.

Check out the result for a few small cases, e.g. the numbers from 1 to 8. Can you start to see patterns which convince you of the general truth of the result?