Car Goat Goat

One of the most successful problems I have used with high school kids is the now notorious "car and two goats" problem. Its fascinating history certainly accounts, in some measure, for its appeal.

Here's the problem. A (female) contestant on a game show, is shown three doors by the (male) host. Behind one of these there is a car, and behind each of the other two there is a goat. She chooses one of the doors (hoping, of course, to get the car) and then, before it is opened, the host (who knows what's behind each door) opens one of the remaining two doors to reveal a goat. At this point he offers her the chance to switch her choice if she so wishes. Should she switch, and if so, how does that change her probability of winning the car?

The problem never fails to produce much heated discussion and disagreement in the classroom. It is clear to everyone that when she makes her initial choice, she has probability 1/3 of selecting the car. But then what is the effect of the host's exposure of one of the goats? Well, my students have given me at least three "answers" to this question.

One answer is that nothing's changed—so whether she switches or not, the probability remains at 1/3. Indeed (so the argument goes) whether her original choice is the car or a goat, the host can always select a door that contains a goat, so no new information is given, and she has no reason to switch. A second answer is that she is now looking at two closed doors, one containing a car and the other containing a goat, so her chances of getting the car are now 1/2, and again that holds whether she switches or not. A third answer is that if she sticks with her original choice, she still has probability 1/3 of winning, but since the car must lie either behind this door or behind the other unopened door, the probability that it lies behind the latter must be 2/3. So she should switch, and she thereby increase her probability of winning the car to 2/3.

And no doubt there are other arguments as well.

This problem received national attention in the United States and Canada when it was posed and solved in the Sept. 9, 1990 issue of Parade Magazine in the "Ask Marilyn" column. The simple solution, given in that issue by the columnist, Marilyn vos Savant (who is listed in the Guinness Book of World Records Hall of Fame as having the "Highest IQ"), was hotly and even sarcastically rebutted by scores of mathematicians in subsequent issues, who claimed that Marilyn was wrong, was out of her depth, and was an amateur trespassing onto professional terrain. The fascinating debate between Marilyn, school teachers, and mathematicians continued for over a year (Parade, Dec. 2, 1990; Feb 17, 1991; July 7, 1991).

In fact, the highlight of the class is not typically when the problem has been solved, but when I then go on to read the many letters that Marilyn published, eminent mathematicians scolding her for being wrong, and elementary school teachers having their class do simple experiments to discover that she's right. It's really quite awesome. Read on!

It's interesting watching the various camps fight it out up at the board. Needless to say, there are a number of questions of clarification that get asked. One thing that has to be made clear is the structure of the game. It's important to assume that the host will always behave in the above manner whether the contestant originally chose a goat or the car. That is, no matter what door the contestant chooses, at least one of the two remaining doors will have a goat, and the host will *always* open such a door, and offer her the chance to change. That's important!—for example if the host's strategy was to give her the chance to switch only when she had originally chosen the car, this would certainly change things. It's likely that some of the original controversy arose because this was not clearly stated in Marilyn's formulation of the problem.

There may be lots of fuzzy arguments presented, and lots of debate, but mathematics has a wonderful advantage over subjects like philosophy, in that there's always (well, almost always) a way to settle the matter with care and precision. For simple probability problems of this type, the analysis can be done with a careful enumeration of the cases.

In order to "set the class up" for a careful analysis, I have them simulate the game. It's impressive how quickly this gives them an accurate understanding of the problem, and a direct path to the solution. I put them in pairs, the contestant who is given a die, and the host who is given a coin. Each pair plays a number of bouts with the non-switching strategy and the same number with the switching strategy. And they keep track of what was won each time.

We'll call the car C and the goats G1 and G2. The contestant starts by throwing the die to determine her initial choice. The scheme at the right makes each door equally likely. The host then chooses a door to open. In the first two cases, his choice is clear (the other goat) and in the last case, he flips a coin, Heads for G1 and Tails for G2.

Die outcome	*Initial choice*
1,2	G1
3,4	G2
5,6	C

Now we look at the results of each strategy.

Not switch. One thing that becomes extremely clear to the students after they've played the game a few times is that if you don't switch, then you will win a car *precisely when you roll a 5 or a 6* and that will of course happen with probability 1/3. Thus, if the contestant uses the non-switching strategy, she will win the car with probability 1/3.

Some students have a lot of trouble accepting this. There is the real feeling that the opening of the door gives you some information that will change your probabilities even if you don't switch. *It's a strange one*

Switch. The second thing that becomes clear to the students is that if all we are interested in is whether you WIN or LOSE the car, the outcome of the coin flip (when it is needed) is irrelevant—the coin only determines which of the two goats you will get if you switch. The final outcome (WIN or LOSE) is entirely determined by the original roll of the die according to the table at the right—if you got a goat on the first roll (first two rows, prob. 2/3)), you win by switching, and if you got a car (last row prob. 1/3), you lose by switching.

roll	*first choice*	*probability*	*outcome of switch*
1,2	G_1	1/3	WIN
3,4	G_2	1/3	WIN
5,6	C	1/3	LOSE

So 2/3 of the time, she will win the car by switching, and hence *the answer to the problem is that she should switch, and that will double her probability of winning to 2/3*.

The letters. Now that happens to be the answer that Marilyn published in the original article. And that was the answer that so many mathematicians objected to. Here are some sample letters that Marilyn received. I will report only the affiliation of the writer.

Now these mathematicians weren't exactly wrong because the original formulation of the problem was somewhat open—e.g. it wasn't made clear that the host would open a door no matter which door the contestant originally picked. But they were certainly very very guilty of sloppy thinking.

University of Florida. You blew it, and you blew it big! I'll explain: After the host reveals a goat, you now have a one-in-two chance of being correct. Whether you change your answer or not, the odds are the same. There is enough mathematical illiteracy in this country, and we don't need the world's highest IQ propagating more. Shame!

George Mason University, Fairfax, Va. Let me explain: If one door is shown to be a loser, that information changes the probabilty to 1/2. As a professional mathematician, I'm very concerned with the general public's lack of mathematical skills. Please help by confessing your error and, in the future, being more careful.

California Faculty Association Your answer to the question is in error. But if it is any consolation, many of my academic colleagues also have been stumped by this problem.

University of Michigan You are in error—and you have ignored good counsel—but Albert Einstein earned a dearer place in the hearts of the people after he admitted his errors.

Millikin University I have been a faithful reader of your column and have not, until now, had any reason to doubt you. However, in this matter, in which I do have expertise, your answer is clearly at odds with the truth.

University of Florida May I suggest that you obtain and refer to a standard textbook on probability before you try to answer a question of this type again?

Georgia State University Your logic is in error, and I am sure you will receive many letters on this topic from high school and college students. Perhaps you should keep a few addresses for help with future columns.

Georgetown University You are utterly incorrect about the game-show question, and I hope this controversy will call some public attention to the serious national crisis in mathematical education. If you can admit your error, you will have contributed constructively toward the solution of a deplorable situation. How many irate mathematicians are needed to get you to change your mind?

Dickinson State University I am in shock that after being corrected by at least three mathematicians, you still do not see your mistake.

Sunriver, Ore. Maybe women look at math problems differently than men.

Western State College You are the goat!

U.S. Army Research Institute You're wrong, but look at the positive side. If all those Ph.D.s were wrong, the country would be in very serious trouble.

Marilyn replies*: Gasp! If this controversy continues, even the postman won't be able to fit into the mailroom. I'm receiving thousands of letter, nearly all insisting that I'm wrong, including one from the deputy director of the Center for Defense Information and another from a research mathematical statistician from the National Institute of Health! Of the letters from the general public, 92% are against my answer, and of the letters from universities, 65% are against my answer. Overall, nine out of 10 readers completely disagree with my reply.*

Marilyn does point out that she received a number of favorable replies. For example:

Massachusetts Inst. of Technology You are indeed correct. My colleagues at work had a ball with this problem, and I dare say that most of them—including me at first—thought you were wrong.

What Marilyn did next was a masterstroke. *As this problem is of such intense interest, I'm willing to put my thinking to the test with a nation-wide experiment. This is a call to math classes all across the country. Set up a probability trial exactly as outlined below…*" And Marilyn went on to describe essentially the dice experiment that we used above.

Over the next weeks a new brand of letter started flowing in, most of these from elementary and high school classes.

Ascension School Chesterfield, Mo. My eighth-grade classes tried it [switching and not switching, 200 times each, using three cups and a coin]. I don't really understand how to set up an equation for your theory, but it definitely does work! You'll have to help rewrite the chapters on probability.

Henry Grady Elementary Tampa, Fla. Our class, with unbridled enthusiasm, is proud to announce that our data support your position. Thank you so much for your faith in America's educators to solve this.

Park View School Wheeling, W.Va. My class had a great time watching your theory come to life. I wish you could have been here to witness it. Their joy is what makes teaching worthwhile.

Webster Elementary School St. Paul, Minn. Seven groups worked on the probability problem. The numbers were impressive, and the students were astounded.

Ridge High School Basking Ridge, N.J. The best part was seeing the looks on the students' faces as their numbers were tallied. The results were thrilling!

Magnolia School Oakdale, Calif. You could hear the kids gasp, one at a time, "Oh my gosh! She was right!"

Westside Elementary River Falls, Wis. I must admit I doubted you until my fifth-grade math class proved you right. All I can say is WOW!

Hebron Public School Hebron, Neb. My classes enjoyed this and look forward to the next project you give America's students. This is the stuff of real science.

Occasionally a student will come up at the end of the class and say, well I followed the solution on the board and I know it was correct and all that, but I still have trouble seeing why switching should give you any advantage. Then here's an enlightening variation. Suppose there are 100 doors—with 1 car and 99 goats, and after you've made your choice, the host opens 98 doors revealing 98 goats. Would you switch? And if so, is your chance of winning still 1 in 100? Does this help your understanding?.

Mabelle Avery School Somers, Conn. Thanks for that fun math problem. I really enjoyed it. It got me out of fractions for two days! Have any more?

Holy Spirit School Annandale, Va. I did your experiment on probability as part of a science-fair project, and after extensive interviews with the judges, I was awarded first place.

And then the professional tide turned.

An M.D. from West Palm Beach, Fla. I also thought you were wrong, so I did your experiment, and you were exactly correct. (I used three cups to represent the three doors, but instead of a penny, I chose an aspirin tablet because I thought I might need to take it after my experiment.)

U.S. Naval Academy Annapolis, Md. I put my solution of the problem on the bulletin board in the Physics Department office here, following it with a declaration that you were right. All morning I took a lot of criticism and abuse from my colleagues, but by late in the afternoon most of them came around. I even won a free dinner from one overconfident professor.

Los Alamos National Laboratory After considerable discussion and vacillation here at the Los Alamos National Laboratory, two of my colleagues independently programmed the problem, and in one million trials, switching paid off 66.7% of the time. The total running time on the computer was less than one second.

San Jose, Calif. Now fess up. Did you really figure all this out, or did you get help from a mathematician?

Since that time a number of mathematical articles have appeared, analyzing the problem and the debate. Two interesting pre-Marilyn articles are worth citing. One in Scientific American 1959 is cited in Problem 2, and the other in 1975 in The American Statistician, 29 (No.1) p.67, refers to the Monte Hall TV show, Let's Make a Deal, which is the setting of Marilyn's problem.

I am grateful to Ed Barbeau for sharing his "Marilyn" file with me.

Problems

1. The following games are played with 8 cards: the four kings and the four aces. The dealer fans the cards with their back to you and asks you to take one, but not to look at it. He tells you that you will win a prize if your chosen card is an ace. There are two versions of what happens next.

(a) In version 1, he closes up the remaining 7 cards, shuffles them and deals the top card face up on the table, and it is seen to be the king of spades. He then deals the next card face down on the table. He then offers you the chance to trade the card you hold for the face-down card that was just dealt. So you have two choices: stay or switch. What are your chances of winning under each choice?

(b) In version 2, he announces that he is going to select a king from the remaining 7 cards, and he selects the king of spades and lays it face up on the table. He then shuffles the remaining six cards and deals the top card face down on the table, and offers you the chance to trade the card you hold for this new card. So you again have two choices: stay or switch. What are your chances of winning under each choice?

2. ***The condemned prisoners.*** In October 1959, Martin Gardiner published the following problem in *Scientific American*. Three men A, B and C are in separate cells under a death sentence. One of these is to be reprieved, but the announcement is not yet to be made. Prisoner A tries to extract the information from the warden, but to no avail. "Then at least," he says, "give me the name of one of the other two who will be executed--that can't hurt, as I can't communicate with them. If B is to be pardoned, give me C's name; if C is to be pardoned, give me B's name; and if I am to be pardoned, give me either B or C by flipping a coin." The warden went off to think about it, and returned to announce that B would be executed. After he left, A smiled to himself at the warden's stupidity--one of A and C would be spared; thus his chances of being spared had just risen to 1/2. The warden also did not know that A had a way of communicating with C by tapping on the wall, and A immediately shared the news with C who was overjoyed to discover that his chances of survival had also risen to 1/2. Did the two men reason correctly?

3. I throw a sequence of 3 darts at a dartboard, aiming for the centre. Assume each one of my throws is equally skillful—that is, I don't have a tendency to get better (or worse!) with each throw. Suppose my second throw is better than my first. What is the probability that my third throw will be better than my second? The next question is the one that really interests me: is this problem really the **car goat goat** problem in disguise? If so explain carefully why these two problems are really the same. If not, explain how they are different.