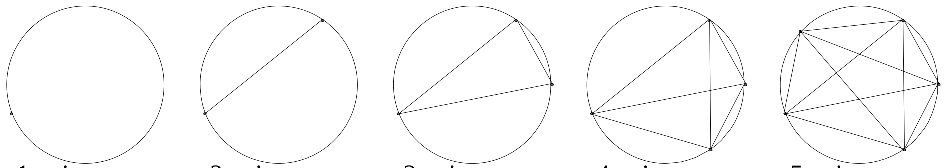
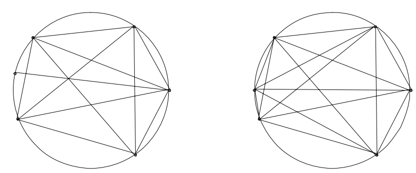
Regions in a circle



1 point	2 points	3 points	4 points	5 points
1 region	2 regions	4 regions	8 regions	16 regions

The question is, what is the next picture? How many regions will 6 points give? There's an obvious guess, of course, and if I take a vote—how many think the answer's 32?—I can be pretty sure of a number of hands. But some of them are getting gun-shy and they sense a trap. And then of course, there are those who never raise their hands on principle.

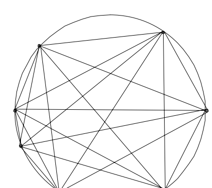
I state the general problem as follows: if I put n points at random on a circle, and join all possible pairs of points, how many regions will I have inside the circle? The pictures above give the answer for n equal to 1,2,3,4, and 5. What is the answer for n=6? For general n?



Not acceptable

acceptable
6 points
31 regions

An explanation is needed for the phrase "at random". The point is that I'm not allowed any coincidences—to be precise, no three lines should ever pass through a common point. Thus, to get the next picture in the above sequence, there are certain places where I'm not allowed to put the 6th point. For example, the picture on the right is acceptable, the one on the left is not. In the picture on the left, one of the new lines generated by the sixth point passed through a previous point of intersection. By moving the sixth point slightly, we get an additional region (the picture on the right). Thus the "triple intersection" robbed us of a region. Another way to impose the "random" condition is to say that the points must be placed to get the maximum number of regions.



7 points
57 regions

The picture above right gives the answer 31 regions for n=6—not 32 but 31! If we add a 7th point, and count very carefully, we get 57 regions. After that, the counting becomes tricky, and we wonder what the general relationship might be.

This is quite a rich problem, and there are many ways to attack it, some sophisticated, and some quite elementary. Once the 57 has been discovered, I find that many of my students hit "the numbers", thinking that if they find the pattern numerically, they will have solved the problem.

The first thing they do is a fast "successive differences" on the *R*-values and they get the table at the right. The last column is certainly compelling. What if that pattern continues? What if the next entry in that column is a 5? I leave it to you to show that the next *R*-value would then be $R(8)=99$. Could this be correct? What sort of argument would we need to establish that? Perhaps we might try to relate the numbers in the difference table to the geometry of the lines and regions, so that we could "see" the pattern in the table not just numerically, but geometrically.

n	*R*			
1	1			
		1		
2	2		1	
		2		1
3	4		2	
		4		2
4	8		4	
		8		3
5	16		7	
		15		4
6	31		11	
		26		
7	57			

The last column of the table is certainly nice. But one of the students notes that we've seen the third column before—in ***cuttingwith planes****. The sequence 1, 2, 4, 8, 15, 26 gives the number of regions when 3-space is cut by 0, 1, 2, 3 etc. planes in general position! Holy cow!*
[See problem 3]

Okay. We are interested in "differences" in successive *R* values, so let's ask what happens to *R* when we add a new point. How many new regions do we get?

Well, each new point necessitates the drawing of a number of new lines, and each of these new lines will create a number of new regions. But how many?

The key observation (which was also the idea behind our analysis of the **cutting with planes** problem) is that this new line creates new regions by slicing through old regions and cutting them in two, and the number of new regions created is the number of old regions this line slices! And a nice way to think about that, is to notice that each time the line intersects an existing line, it leaves one old region and enters another, so the number of old regions it encounters is one more than the number of old lines it meets.

We need some notation. We are focusing on lines and intersections of lines, so let's keep track of the number of each at each stage. Let *L* denote the total number of lines, *I* the number of intersections of lines (inside the circle) and *R* as before the number of regions. If we tabulate these quantities for the first seven pictures, we get the table at the right.

n	*L*	*I*	*R*
1	0	0	1
2	1	0	2
3	3	0	4
4	6	1	8
5	10	5	16
6	15	15	31
7	21	35	57

I ask the class to find patterns in the table, and immediately we get the observation that:

$$R = 1 + L + I.$$

Hmm. This is nice, because if it's true, then we can obtain the *R*-column by figuring out the (possibly simpler) *L*-column and *I*-column.

But why should the above equation hold? Again we have a numerical observation that we would like to verify with a geometric argument.

Well, the above argument for the creation of new regions provides an inductive type of proof! Indeed, what happens when we draw a new line? Well, R and I both increase, and what we argued above was that the increase in R will be one more than the increase in I. Now since L also increases by 1, if the formula held before the drawing of the new line, it must still hold afterwards. To emphasize this, if the new line meets 12 old lines, then I will have increased by 12, L will have increased by 1, and R will have increased by 13—and the equation will remain balanced. Formally, what we have here is a proof by induction on n, the number of points. First we note that the formula holds for n=1 (in that case, R=1, I=0 and L=0). Then note that to move from any n to n+1 we add a number of lines, and the above argument shows that the formula continues to hold with the addition of each new line.

Having perceived and established this formula, our attention now shifts from R to L and I (which ought to be simpler), and we now try to get hold of them.

n	*L*	*I*	*R*
1	0	0	1
2	1	0	2
3	3	0	4
4	6	1	8
5	10	5	16
6	15	15	31
7	21	35	57

The L-column is simple enough. The students note quite quickly that the successive differences in the L-column are 1,2,3,4, etc. Of course, it is clear why this holds: consider the addition of the 6th point. This will necessitate the drawing of 5 new lines (to each existing point). So L must increase by 5. In general, the addition of the nth point will increase L by n–1. By adding up the differences, we even get a *formula* for L in terms of n:

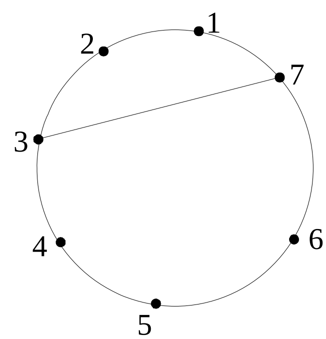
$$L(n) = 1 + 2 + 3 + ... + (n-1) = \frac{(n-1)n}{2}.$$

The expression on the right comes from a standard formula for the sum of the first n natural numbers. See problem 2.

n	*I*			
1	0			
		0		
2	0		0	
		0		1
3	0		1	
		1		2
4	1		3	
		4		3
5	5		6	
		10		4
6	15		10	
		20		
7	35			

The I-pattern is a bit more interesting, and I put the class into small group discussion mode for a time. A number of arguments emerge. A popular approach is to do successive differences again and we get a similar pattern to what happened when we tried this with R. But this time one of the groups gives me an “anatomy” of the first-difference column: 0,0,1,4,10,20.

Consider the 20. That represents the number of intersections created by the addition of the 7^{th} point. Now these intersections come from joining the 7^{th} point to the existing points. For example, consider the effect of joining the 7^{th} point to the 3^{rd} point. That new line will intersect all the lines that connect points on one side of the line to points on the other. Now there are 2 points on one side and 3 points on the other for a total of $2\times3=6$ possible lines and therefore $2\times3=6$ new intersections. Similarly, the effect of joining the 7^{th} point to the 2nd point will be to create a total of $1\times4=4$ new intersections. The total number of new intersections due to the addition of the 7^{th} point is:



$$0\times5 + 1\times4 + 2\times3 + 3\times2 + 4\times1 + 5\times0 = 20$$

and that's an "anatomy" of the 20. We can of course omit the first and the last terms which will always give 0.

We get a similar anatomy of the other entries in the column. For example the 10 is:

$$1\times3 + 2\times2 + 3\times1 = 10.$$

This analysis makes it easy to generate the I column entry-by-entry and if we put this together with our L-formula, this gives us a simple numerical algorithm for generating successive values of R.

For example, to get the value of $R(8)$ we calculate:

$$L(8) = \frac{(n-1)n}{2} = \frac{7\times8}{2} = 28$$

$$I(8) = I(7) + (1\times5 + 2\times4 + 3\times3 + 4\times2 + 5\times1)$$

$$= 35 + 35 = 70$$

$$R(8) = 1 + L(8) + I(8) = 1 + 28 + 70 = 99.$$

n	L	I	R
1	0	0	1
2	1	0	2
3	3	0	4
4	6	1	8
5	10	5	16
6	15	15	31
7	21	35	57
8	28	70	99

This method is a great advance on counting regions on a picture, and we should be proud of ourselves!

Of course, this is no way to get the R value for n=100 (except with a computer). The point is that, while we have a general formula for L, we don't have one for I; all we have is a recursive arithmetical "machine". Can we in fact get a formula for I?

The answer is yes, and the above difference scheme can be used to get it; the procedure is just a bit more complicated than the argument that worked for the L-formula. But I'm not going to follow that up right now either because there's something else around—something quite extraordinary...

Some years ago I was discussing this problem with a high school class, and one of the students was a Pascal triangle nut (someone who believes that everything good in life is somehow hidden in Pascal's triangle). So while we were busy writing down bigger and bigger R tables, he was writing down Pascal's triangle. And what he saw there was quite astonishing—the L column is the second diagonal and the I column is the fourth diagonal! NOTE: there's an unexpected counting convention here—we call the extreme left diagonal of all 1's the "zeroth" diagonal. In fact, that zeroth diagonal could just represent the "1" which is the first term in the R formula.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}$$

Isn't that something? What on earth are we to make of it? The first thing to say is that, if this pattern does indeed hold, it gives us a nice formula for R because of course we have a combinatorial interpretation of each of the entries in Pascal's triangle. Indeed, the kth entry (from the left) in the nth row (where the 1 at the top is taken to be the 0th row) is the number of ways of choosing k objects from n [see **Pascal**] which is written $\binom{n}{k}$. For example, there are 15 ways of choosing 3 objects from 6, so that $\binom{6}{3} = 15$. This gives us very elegant formulae for L, I and therefore R, and these are displayed at the right.

$$L = \binom{n}{2}$$

$$I = \binom{n}{4}$$

$$R = 1 + L + I = \binom{n}{0} + \binom{n}{2} + \binom{n}{4}$$

Below, we use these formulae to get the next entries of our R-table, for n=9:

$$L(9) = \binom{9}{2} = \frac{9 \cdot 8}{2 \cdot 1} = 36$$

$$I(9) = \binom{9}{4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 126$$

$$R(9) = \binom{9}{0} + \binom{9}{2} + \binom{9}{4} = 1 + 36 + 126 = 163$$

Of course, the question is, why should these combinatorial formulae hold? They are so simple and elegant, one feels that there must be an interpretation of L and I in terms of choosing objects from sets, that makes these formulae clear. Can we find a combinatorial interpretation of L and I?

Well L is not so hard: with n points, you get one line for every pair of points, so L is the number of ways of choosing 2 things from n. The argument for I is more subtle, but just as simple: notice that there is a 1-1 correspondence between interior intersections and sets of 4 points on the circle. Every set of four points on the circle determines exactly one intersection, and every intersection is accounted for in this way. So the total number of intersection I is just the number of ways of choosing 4 things from n. Wow.

Problems

1. Look at the column of first differences of I: 0,0,1,4,10,20 (see table at the right). Identify this as a diagonal of Pascal's triangle (given above), and thus write each of these numbers as a combinatorial coefficient. For example, 20 gets identified as $\binom{5}{3}$. Argue on geometric grounds that that's exactly what it should be.

n	I			
1	0			
		0		
2	0		0	
		0		1
3	0		1	
		1		2
4	1		3	
		4		3
5	5		6	
		10		4
6	15		10	
		20		
7	35			

2. *The formula for the sum of the numbers from 1 to n.*
Write out the numbers between 1 and 9. Argue on grounds of symmetry that their *average* should be 5. Then argue that their *sum* should be 9 times their average which is 45. Use this same idea to show that the sum of the numbers from 1 to n is given by the formula:

$$1+2+3+...+n=\frac{n(n+1)}{2}.$$

3. Return to **cutting with planes**. While we are in the "elegant formula" business, a number of students noticed the similarity between the above R numbers and some of the number we had generated previously looking at the number of regions created by n planes. To be precise, the sequence 1, 2, 4, 8, 16, 31, 57, etc. turns out to give the number of regions of 4-space created by cutting it with 0, 1, 2, 3, 4, 5, and 6 hyperplanes, respectively (**cutting with planes** Problem 4). That is, n points in the above circle problem "corresponds" to n–1 hyperplanes in the 4-space cutting problem. That suggests that the above R-formula can also be used to give the number of regions created by n hyperplanes in 4-space. It will be

$$R=\binom{n+1}{0}+\binom{n+1}{2}+\binom{n+1}{4}.$$

We have to use n+1 instead of n because, for example, the R value for the case of 5 hyperplanes seems to correspond to the R-value for 6 points on the circle. Actually, it would be nice to have a formula which had n in place of n+1, and in fact we can get that from the rule: $\binom{n+1}{r}=\binom{n}{r-1}+\binom{n}{r}$. This is in fact the "addition rule" which is used to generate Pascal's triangle. [See **Pascal**]. Using this (for r>0) we get a spectacular formula for the number of regions created by:

n hyperplanes in 4-space: $R=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\binom{n}{4}.$

Why stop there?

n planes in 3-space: $R=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}$

n lines in 2-space: $R=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}$

n points in 1-space: $R=\binom{n}{0}+\binom{n}{1}$

Another wow.
Is there any geometric logic behind these decompositions?