Coffee and cream

I was at a math teacher's workshop a few years ago and we were trying to teach ourselves how to orchestrate a curriculum that was based, not on the simple transmission of skills, but on exploratory problem-solving.
At one point we were trying to find a good problem to spend a hour playing with, and so various people suggested problems or at least scenarios they had encountered, and one of these was the coffee and cream problem and for a reason I don't fully understand, that problem really caught the imagination of the group. Hard to say why. Maybe it's because coffee is such a great drink. And maybe because it's just a bit whimsical.

Anyway it <u>is</u> a good exploratory problem. There are some basic physical principles at work here but they are ones that the students can pretty well construct based on their intuition (and with a little help from the front). And there's some basic and central math too— exponential decay and weighted average.

It comes in many different versions, but here's a simple one. I buy my large coffee and it's in one of these heavy cardboard cups with a plastic top, and I pick up a couple of creamers, and I'm out to hit the road. Now I won't be able to settle down to drink the coffee till I get on the highway, otherwise it's too much "start and stop." So it's going to be 10 minutes before I can drink it. And I'm one of those who has to have my coffee hot, at least as hot as possible. So I want it to have maximum possible temperature when I come to drink it. And the question is—do you put the cream in when you buy the coffee or after ten minutes when you're ready to drink it?

Problem. I buy a large coffee (300 ml) at 80° and a couple of creamers (15 ml each) at 4°, and get into my car which stays at a constant temperature of 16°. It will be 10 minutes before I can drink the coffee, and I want it to be as hot as possible at that time. Do I put the cream in at the beginning or after 10 minutes when I'm ready to drink it?

Mix last. Let's start by keeping them separate till the end. Then what will happen to each during this 10-minute period? Well the coffee will cool down. Why?—because it's hotter than the car. And the cream will warm up because it's cooler than the car. Okay. It seems we'll need to know something about just how fast those two processes occur. Can we get a bit more information here?

Sure. Let me tell you precisely what happens to the temperature of each. After 10 minutes in the car, the coffee drops from 80° to 76° and the cream rises from 4° to 7°.

Well that's certainly a good piece of information. In fact it ought to enable us to figure out the temperature of the mixture when, at the end, we put them together. Let's see, the coffee's 76°, the cream's 7°—what will the mixture be—the average of the two? Not quite. There's more coffee than cream, so that should have more weight. How much more weight? Maybe the weights should be proportional to the amounts. In fact that's it exactly: when you mix two liquids at different temperatures, the final temperature is the weighted average of the two where the weights to use are the amounts.

Thus, given 300 ml coffee at 76° and 30 ml cream at 7°, the final temperature for the "mix at the end" strategy is the weighted average:

mix last: $$T = \frac{300(76)+30(7)}{330} = \frac{23010}{330} = 69.7\,.$$

> Here's the principle. If liquid 1 has temperature T_1 and volume v_1 and liquid 2 has temperature T_2 and volume v_2, then when they are mixed the resulting temperature will be the weighted average:
> $$T = \frac{v_1 T_1 + v_2 T_2}{v_1 + v_2}\,.$$
> *BUT* this only holds when the liquids are the same or at least have the same specific heat. [See problem 4.] For our purposes, coffee and cream are both mainly water, so we treat them as being the same substance.

Mix first. If we mix at the beginning, we'll right away get the weighted average but now using the starting temperatures 80° and 4°:

$$T = \frac{300(80)+30(4)}{330} = \frac{24120}{330} = 73.1\,.$$

So this is the temperature of the mixture at the beginning. Now the question is, what happens to it over the 10 minutes it sits in the car?

Hmm. The only temperature change information we have so far pertains to the coffee and cream separately. Is that of any help to us once we mix them?

To penetrate this, we have to understand the law of temperature change (see Box below: Newton's Law of Cooling). A consequence of this law is that the temperature *difference* $D = T–16$ decreases at a constant *percentage* rate. And that makes D an exponentially decaying function:

$$D = D_0\, r^t.$$

where D_0 is the value of D at the beginning (t=0) and r is the factor by which the difference is multiplied in each minute. For example, if r = 0.98, then the difference D loses 2% every minute. Clearly r will be determined by the nature of the interface between the object and the environment. If the object is well insulated, r will be close to 1, if the object is more "exposed," r will be smaller.

Law of temperature change.

Suppose a hot object sits in a cool environment held at constant temperature *E*. Then the object will cool down at a rate which is proportional to the difference in temperature between object and environment.

This is a fundamental principle known as Newton's Law of Cooling. This Law also holds for cool objects in a warmer environment.

Now r is the "one-minute" multiplier but of course we will have a multiplier belonging to *any* time interval—for example the 10-minute multiplier is r^{10}. Now the coffee-mug data we are given belongs to a 10-minute interval. In 10 minutes the temperature T falls from 80 to 76, and that tells us that the difference D falls from 64 to 60 (Table). That gives us a 10-minute multiplier of 60/64 = 15/16.

Unmixed coffee		
t	*T*	*D*=*T*–16
0	80	64
10	76	60

Does this help us determine what happens to the coffee-cream mixture over the same 10 minutes? Yes it does. When we mix the two liquids, we put the cream into the coffee, not the other way 'round, so *the container we are using is the original coffee container and the multiplier is determined by that container*. That means the same multiplier, 15/16, ought to apply.

Apply this multiplier to the mixed coffee. It starts at 73.1° (calculated above). That gives a difference of 57.1° (Table). In 10 minutes that difference gets multiplied by 15/16 giving us:

$$\frac{15}{16}57.1 = 53.5$$

Mixed coffee		
t	*T*	*D*=*T*–16
0	73.1	57.1
10	69.5	53.5

and from that we calculate the temperature to be:

mix first: $T = 16 + 53.5 = 69.5.$

That's the answer It's very slightly smaller than the mix-last case. It looks like you're better to keep them separate till you're ready to drink.

"All that work for one fifth of a degree!?"
Isn't math wonderful.

We have two final temperatures:

mix last:	69.7°
mix first:	69.5°

They differ by a small amount. My objective now is to try to understand *exactly where this difference comes from*. I want an "analysis" of the two numbers which will reveal the source of the difference, so that I could say, well if I changed this or that then I'd get the opposite result, that it would be better to mix at the beginning. Somehow, I want to "take those two numbers apart."

In thinking a bit about the best way to do that, I am enticed by the simplicity of the exponential law for the temperature difference, and that should clearly be at the center of the analysis. It fact it strikes me that it would be nice to have to deal only with the temperature *difference* and not with the temperature itself.

Well why not?—D carries all the information we need, in that to get T from it we need only add 16. So let's work with D. If you like, we are just rescaling the thermometer—changing 16 to 0, 17 to 1, 18 to 2, etc.

starting values		
	T	$D=T-16$
coffee	80	64
cream	4	–12

Mix last. The coffee starts with D-value 64 and the cream with D-value –12. Over the 10-minute period, these are multiplied by the 10-minute multipliers. We have already found the multiplier for the coffee container: 15/16. But what about the cream container? Well in 10 minutes the cream difference goes from –12 to –9, that's a multiplier of 3/4. To have a "clean" expression, let's give names to these two multipliers. We'll call the coffee multiplier $\alpha = 15/16$ and the cream multiplier $\beta = 3/4$.

coffee		
t	T	$D=T-16$
0	80	64
10	76	60

cream		
t	T	$D=T-16$
0	4	–12
10	7	–9

At the end of the 10-minute period, we mix them giving us a final difference which is the weighted average:

mix last:
$$D = \frac{300(60)+30(-9)}{330} = \frac{300(64\alpha)-30(12\beta)}{330}.$$

Mix first. In this case we take the weighted average at the beginning and *then* we apply the 10-minute multiplier. And the multiplier to use is the coffee one, α, because the mixture is put into the coffee container.

mix first:
$$D = \frac{300(64)+30(-12)}{330}\alpha = \frac{300(64\alpha)-30(12\alpha)}{330}.$$

Now compare the two final expressions. The only difference is that the β in the first becomes α in the second. Since $\beta<\alpha$, and the last term is subtracted, and we want D to be as large as possible, we take the first expression—mix last.

These two expressions are easily compared. To get the largest D we simply need to know which of α and β is the larger. In this case, α is bigger reflecting the fact that the heavy cardboard coffee cup is more insulated than the plastic cream container.

So that's the real reason it's better to mix first—*it's a question of where the cream should spend the 10 minutes*. The answer is that since the car is warmer than the cream, it should be in the container that's least insulated—to pick up as much heat as possible. Make sense?

Newton's Law of Cooling.

Take the coffee. It cools down because it's hotter than the car. Let's be more precise—the *rate* at which it cools depends on the *difference* in temperature between the coffee and the car. [Right?—if there was no difference, there'd be no change.] And the greater this difference, the faster is the change. In fact the precise relationship between the temperature difference and the rate of change is as simple as it gets—they're proportional—half the temperature difference means half the rate of change.

Let's have some notation: Let the temperature of the coffee be T and the temperature *difference* with the air be $D = T–E$. [In our example, E = 16, but here we'll just call it E. What's important is that it remains constant.] The law says that T decreases at a rate proportional to D. Now since T and D decrease at the same rate (since E is constant) we conclude that *D decreases at a rate proportional to D.* That is, D decreases at a constant *percentage* rate. Well that makes D an exponentially decaying function:

$$D = D_0 r^t$$

where r, the 1-minute multiplier, will be determined by the nature of the interface between the object and the environment. If the object is well insulated, r will be close to 1, if the object is more "exposed," r will be smaller.

We can obtain a multiplier for *any* time interval. For example, over any 5-minute period, D will be multiplied by r^5 so that this is called "the 5-minute multiplier."

Example 1. Suppose a hot object sits in a room held at a fixed temperature of 20°. Suppose that at a certain moment (t=0) its temperature is 70° and that 10 minutes later (t=10) its temperature has fallen to 60°. Question: what will be the temperature after another 10 minutes (at t=20)?

Below I have tabulated the temperature T of the object and the temperature difference D with the room. Can you fill in the question marks?

t	T	D=T–20
0	70	50
10	60	40
20	?	?

I can always count on a student to guess that at t=20, we should have T=50 and D=30. That would have both T and D changing *at a constant rate.* But that's *not* what happens—rather the difference D changes at a constant *percentage* rate. That's what a constant *multiplier* means.

So what does the table tell us? In the first 10-minute period, D changes from 50 to 40. The multiplier for that is 40/50 = 4/5 = 0.8, so that's the 10-minute multiplier. Every 10 minutes D is multiplied by 80% (that is, it loses 20%). Thus in the second 10-minute period, D will again be multiplied by 80%. Now 80% of 40 is 32, so $D(20)$ = 32. That gives us a temperature of $T = 20+D = 52$.

t	T	D=T–20
0	70	50
10	60	40
20	52	32

Example 2. For the object of Example 1, what is the temperature T after an hour, at t=60?

Use the 10-minute multiplier, 0.8. Every 10 minutes, D is multiplied by 0.8. Thus after 60 minutes, D is multiplied by 0.8 6 times:

$$D(60) = 50(0.8)^6 = 13.1$$

The temperature is then $T = 20 + D = 20 + 13.1 = 33.1$.

What about t=5? See problem 5.

Problems

1. A 250 ml mug of coffee and a 30 ml container of milk both sit on the counter. The coffee is at 70°, the milk is at 5°, and the room stays at a constant temperature of 20°. If I leave the milk and the coffee standing separately for 5 minutes, the coffee will have fallen to 60° and the milk will have risen to 8°.
(a) Suppose I mix them at this point. What is the temperature of the mixture?
(b) Suppose I mix them at the beginning in the coffee mug, and then let the mixture stand for 5 minutes. What is the temperature of the mixture at this point?

2. I pour myself a cup of tea (250 ml) at 80° and leave it on the counter to cool in a room that stays at the constant temperature of 20°. I want to drink it in 5 minutes and I want it to be as <u>cool</u> as possible. I have two choices: take the milk out of the fridge (at 5°) and pour it in now (25 ml) or wait till I'm ready to drink it, and then take the milk out of the fridge. Calculate the drinking temperature of the mixture under each option. Assume that if the tea is left on its own to stand on the counter, its temperature after 5 minutes will be 74°.

3. I have three steel marbles, A, B and C all at room temperature (20°). I have an oven maintained at 100° and a freezer maintained at –20°. I put A in the oven and B in the freezer for 5 minutes, and then I switch them: B goes into the oven and A into the freezer, again for 5 minutes. Poor C just gets to sit on the counter the whole time. At the end of the 10 minutes, rank the marbles according to temperature.

4. *Heat capacity.* The heat capacity of a substance is the amount of heat energy required to raise the temperature of 1 g of the substance by 1° C. For example, the heat capacity of water is 4.2joules/g°C. That means that to raise the temperature of 1 litre (= 1000 g) of water by 1° C would require 4200 joules—that's the amount of energy used by a 100 watt light bulb in 42 seconds.
(a) Suppose that you have 3 litres of water at 20° and 2 litres of water at 40°. If you mix them, what is the temperature of the mixture? Now you already know how to do this—take a weighted average of the two temperatures. What I want you to do here is use the heat capacity concept to derive this "weighted average" result. The idea is that the amount of heat lost by one part must equal the amount of heat gained by the other. [Hint: let the final temperature be an unknown T.]

(b) Alcohol has a heat capacity of 2.4 joules/g°C. That's smaller than water—you don't have to use so much heat to get the same temperature rise. What does that do to our weighted average formula? Suppose that you have 3 litres of alcohol at 20° and 2 litres of water at 40°. If you mix them, what is the temperature of the mixture?

(c) Is your answer to (b) in any sense a weighted average? Can you provide a general formula for this situation?

5. (Extension of Example 1). Suppose a hot object sits in a room held at a fixed temperature of 20°. Suppose that at a certain moment (t=0) its temperature is 70° and that 10 minutes later (t=10) its temperature has fallen to 60°. Question: what was its temperature after only 5 minutes (at t=5)? After 15 minutes (at t=15)?
[What would the 5-minute multiplier be? What's its relationship with the 10-minute multiplier?]