Cutting a cube in half

Warm up. Take a cube. Attach a string to a vertex and hang it. So the top point is a vertex and the bottom point is the diagonally opposite vertex.

Now take a horizontal plane and slice it exactly in half. Right through the centre.

What does the cross-section look like—the intersection of the plane and the cube?

You should try this problem first without having a cube to look at. And then maybe to do with a cube sitting flat on your desk, but not to be picked up. And then finally pick it up, and you see that the intersection is a hexagon.

Then you might ask why it is a hexagon, and why you might have expected that. [Indeed that's an important question for making progress with the next problem.]

And the answer is that the cube has six identical faces and the intersection plane has to cut each of these faces, and each cut gives us a side (or face) of the hexagon.

Furthermore, the intersection plane has to cut each face *in exactly the same way*, so that the hexagon is *regular*—all its edges are the same length. Before moving on, try to find an argument for that.

An elegant argument uses the symmetry of the cube. [A less insightful argument might actually calculate each intersection algebraically and compare. Though later watch us do just that…] Think of the cube with the cutting plane passing through it. Now rotate the whole configuration about the central vertical axis by 1/3 of a revolution. What happens? Both the cube and the cutting plane transform into themselves but each face of the cube transforms into an adjacent face. *And so the intersection of the cutting plane with each face transforms into its intersection with the adjacent face.* So those two intersections have to be "congruent." Since they're lines they'll have to have the same length.

What that tells us that the top three intersections are congruent and the bottom three intersections are congruent. To get the tops congruent to the bottoms, reflect the whole configuration through the centre point of the cube. Again both the cube and the cutting plane transform into themselves and the top faces transform into the bottom faces. So the top and bottom intersections are the same length.
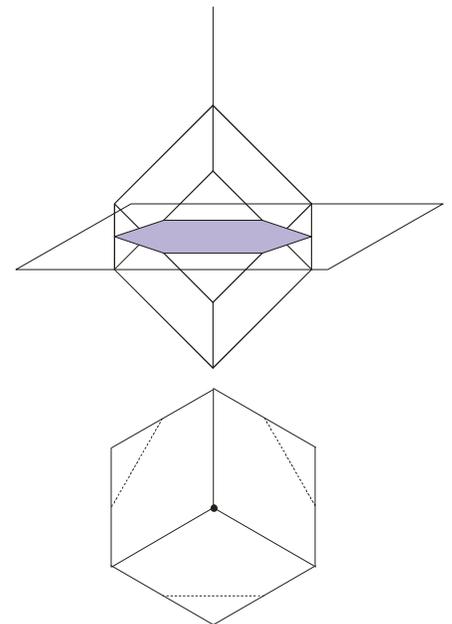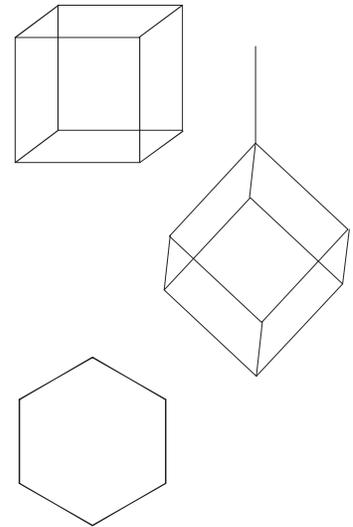
Hanging cube viewed from directly above. The transformation we describe is a rotation through 120º about the centre.

Now for the real problem.
Take a 4-dimensional hypercube. [A tesseract.] Hang it by a vertex. Cut it in half with a horizontal 3-dimensional hyperplane. The intersection is a 3-dimensional polyhedron. You could hold it in your hand. What does it look like? Make me a model.

Algebraic description of the cube
We argue by analogy with the 3-D case but because of the difficulty in visualizing the geometric analogue, we turn to algebra as a more concrete representation. Set the 3-D cube in a coordinate system so we have a set of coordinates for every point. Rather than work with the cube rotated into hanging position, we sit it "flat on the desk" to give ourselves a simple description of the coordinates of the vertices. The cube is the set of all points (x, y, z) for which $0 \leq x, y, z \leq 1$ and the faces of the cube are the planes:

$$x=0 \qquad y=0 \qquad z=0$$
$$x=1 \qquad y=1 \qquad z=1$$

We take the top vertex to be $(1,1,1)$ and then the bottom will be $(0,0,0)$. That choice seems to make sense as it gives a coordinate symmetry $x=y=z$ to the central axis.

Since we're doing algebra, let's describe the cutting plane. It will be perpendicular to the vector $(1,1,1)$ so it will have an equation of the form $x+y+z=k$ and since it must pass through the point $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ which is the midpoint of the diagonal from $(0,0,0)$ to $(1,1,1)$ k must be $3/2$ and the cutting plane is $x+y+z=3/2$.

Now for the hypercube
So now let's move to 4 dimensions. The hypercube is the set of all points (w, x, y, z) for which $0 \leq w, x, y, z \leq 1$ and the faces of the hypercube are the planes:

$$w=0 \qquad x=0 \qquad y=0 \qquad z=0$$
$$w=1 \qquad x=1 \qquad y=1 \qquad z=1$$

Geometrically these are copies of the 3-D cube. For example, fixing $w = 0$ leaves us with the set $0 \leq x, y, z \leq 1$ and this is precisely the 3-D cube.

Again we take the top vertex to be $(1,1,1,1)$ and the bottom will be $(0,0,0,0)$. The midpoint of the diagonal is $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$, so that the cutting hyperplane is $w+x+y+z=2$.

Okay. The hyperplane cuts the hypercube and we want to describe the intersection set. Its faces will be the intersection of the hyperplane with the faces of the hypercube, and we expect these to be 2-dimensional. Let's try to describe them.

This is an awesome problem. More than any I have recently seen it illustrates the power of a simple idea, if used creatively, to circumvent a difficult piece of analysis. In this case, the idea is that of symmetry.
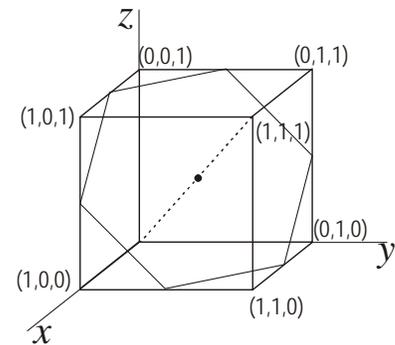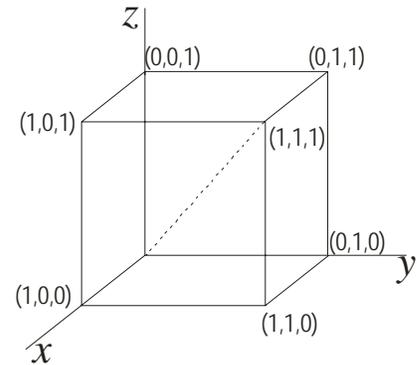

We visualize the hypercube using the 3-D picture above, being careful with our generalizations. The coordinate algebra serves as a critical check on our arguments.

The congruence of the faces of the intersection.
First of all there will be 8 of them, one for each of the eight faces listed above. Let's first try to show that they will all be congruent.

Our 3-D strategy was to rotate the cube so that one face mapped onto another and the cutting plane mapped to itself. Our intuition tells us the same strategy should work in 4-D except those "rotations," or whatever they are, are a bit elusive. It's hard to be sure we can really do it.

One idea is to try to find an alternative algebraic description of the rotation we used for the cube—a description that could be more easily generalized. What does the rotation do to the coordinates of a typical point?
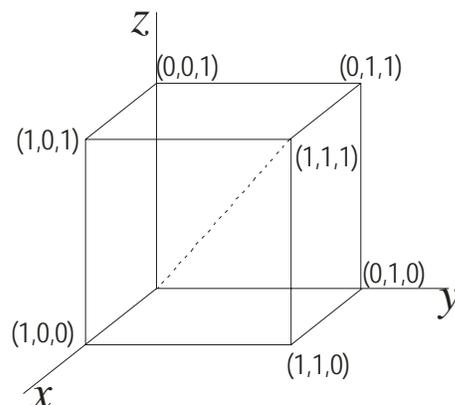
Well look at the vertices—look what happens to them. Consider the counter-clockwise rotation that puts $z=1$ into $x=1$ and puts $x=1$ into $y=1$, etc. You can easily track it in the diagram at the right.
(1,0,1) maps to (1,1,0)
(1,1,0) maps to (0,1,1)
(0,1,1) maps to (1,0,1).
After a few more such observations, you might guess that each time the last coordinate moves into first place. That is:

(a,b,c) maps to (c,a,b).

This is a transformation which is easily seen to be a bijection of the cube onto itself, and to cycle both the top faces and the bottom faces. It also maps the cutting plane to itself as

if $a+b+c=3/2$ then $c+a+b=3/2$.

Well this description is easily generalized to 4-space. The transformation

(a,b,c,d) maps to (d,a,b,c)

is easily seen to be a bijection of the hypercube onto itself, and to cycle both the top faces and the bottom faces (track the vertices). It also maps the cutting plane to itself as if $a+b+c+d = 2$ then also $d+a+b+c = 2$. We conclude that the intersection sets for the top faces are all congruent and the intersection sets for the bottom faces are also all congruent.

It remains to find a transformation which maps the top faces to the bottom faces. That's not hard to guess at: just change 0 into 1 and 1 into 0. For example, (1,1,0,1) will map to (0,0,1,0). Geometrically, this is a reflection through the midpoint (½,½,½,½).

We actually need more to get a geometric congruence here. We need the transformation to preserve geometric form. To do that, it must be *isometric* (preserve distances). And it is.

The geometric shape of the faces of the intersection.
We have shown that the intersection set has 8 congruent faces.
Now we need to figure out what these faces look like.

Again, algebra provides the right toolset. The hypercube is the
set $0 \leq w, x, y, z \leq 1$ and the 8 faces are

$$w=0 \qquad x=0 \qquad y=0 \qquad z=0$$
$$w=1 \qquad x=1 \qquad y=1 \qquad z=1$$

The cutting hyperplane has equation $w+x+y+z=2$.

Let's take one of these faces and see how the hyperplane inter-
sects it. For example take $w = 1$. Then the intersection with
$w+x+y+z=2$ will be the set $x+y+z=1$. Thought of as a subset of
3-space, it's an entire plane. Now we take the subset for which
$0 \leq x, y, z \leq 1$. That's easily seen to be the part inside the positive
octant $x, y, z \geq 0$ and that's easily seen to be an equilateral trian-
gle.

A similar argument works for the faces $x=1$, $y=1$ and $z=1$. So the
top faces are equilateral triangles.

Now take a typical bottom face, $w=0$. Then the intersection with
$w+x+y+z=2$ will be the set $x+y+z=2$. Thought of as a subset of
3-space, it's again an entire plane. But to get hold of the subset
for which $0 \leq x, y, z \leq 1$, we need an argument a bit different from
before.

Fasten attention on the triangle in (x, y, z) space with vertices
$(2,0,0)$, $(0,2,0)$ and $(0,0,2)$. At all points we already have x, y, z
≥ 0, but how do we identify the regions in which $x, y, z \leq 1$?

Well the planes $x=1$, $y=1$ and $z=1$ intersect the triangle in lines
which are half-way between the vertices and the opposite edges.
The three lines so formed form an equilateral triangle. The re-
gion inside that triangle is the set for which x, y and z are all less
than 1.

We have found the shape of the faces of the intersection set.
They are all equilateral triangles of the same size.

Do we know an 8-sided polyhedron whose faces are equilateral
triangles? Yes we do: the regular octahedron, one of the five
Platonic solids.

That's the intersection set we are looking for. If you hang a hy-
percube by a vertex and cut it in half with a horizontal 3-
dimensional hyperplane, the intersection set is an octahedron.

We want the intersection of the
planes $w=1$ and $w+x+y+z=2$, and
then we want the subset of that
defined by
$$0 \leq w, x, y, z \leq 1$$
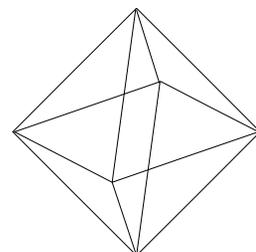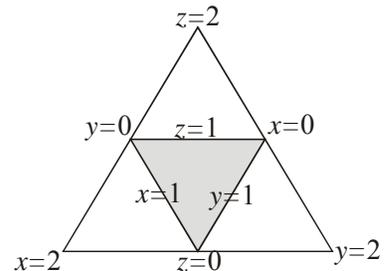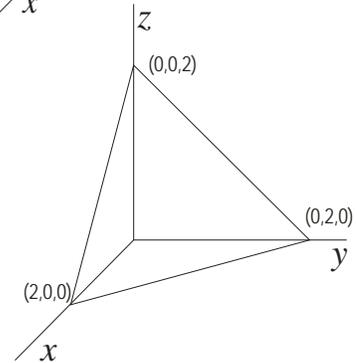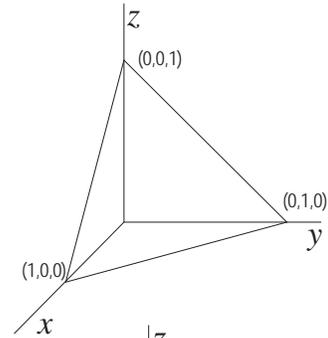
The regular octahedron

Problems

1. Take the octahedron we found above and mark the correct 4-D coordinates on its 6 vertices. Of course there will be many right answers depending on how you decide to orient of the octahedron, but we have to be careful to get internal consistency at each triangular face. Once you have the picture labeled, label each of the 8 triangles with the correct face of the hypercube.
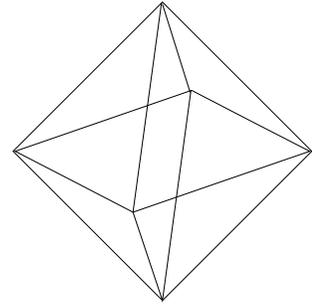
2.(a) Think about the 3-dimensional case in which the cutting plane passed through the centre of the hanging cube giving us a hexagonal cross-section. Suppose we slowly raise the cutting plane. Then we expect the shape of the intersection to change. Finally, when the cutting plane is raised all the way to the top, the intersection will be the single point $(1,1,1)$. Describe how the intersection changes as the plane is lifted.

(b) Now, using the analogy between 3 and 4 dimensions, what can you say about how the intersection changes in the 4-dimensional case?

3. Take a five-dimensional cube, suspend it from a vertex, and run a four-dimensional hyperplane right through the centre. The intersection will be a four-dimensional polytope, and I guess we won't have much success in visualizing it, but I expect it to have all kinds of symmetry, just like its lower-dimensional cousins, and in particular, I expect the "faces" of this polytope to be congruent. Now these faces will be three-dimensional polyhedra. What do they look like? How closely can you describe them? Identify the coordinates of the vertices of a typical such 3-dimensional face.

This is a wonderful problem with a lovely answer. See below.

The Platonic solids. A polyhedron is called regular if it has identical faces which are all regular polygons (all sides the same length). There are exactly five regular polyhedra, the "Platonic solids" and they are the tetrahedron (4 triangular faces), the cube (6 square faces), the octagon (8 triangular faces), the dodecahedron (12 pentagonal faces) and the icosahedron (20 triangular faces).

tetrahedron   cube   octahedron

dodecahedron   icosahedron