Even odd

Eeyore and Owl play the following game. They flip ten coins; Eeyore wins if the number of heads is even, and Owl wins if it's odd. Is the game fair, or does it favour one or the other?

I constructed this problem because I needed something to help us to move essentially from cards to coins. That is, having interpreted the combinatorial coefficient $\binom{n}{k}$ as the number of ways of choosing a hand of k cards from a deck of size n, we now want to think of it as the number of ways of getting k heads in a toss of n coins.

I was surprised at the number of rich directions that it and its eccentric cc

I throw the problem out to the class, and a loose discussion gets under way. Many seem to feel that the game should be fair, that each player should have a 50% chance of winning—but others are doubtful.

Doesn't it follow right from the fact that heads and tails are equally likely—that the game has to be fair?

On the other hand:

The most likely number of heads is 5, and that's an odd number. Doesn't that mean odd is more likely?

How be we look at the situation with a smaller number of coins?

Excellent idea!—I give everyone a moment to find the solution for two coins. Here we can easily list the four possibilities. Note that there are four possibilities and not three. The easiest way to get this right is to number the coins—two possibilities for the first coin, and two for the second.

Two coins	
outcomes	*# heads*
TT	0
TH	1
HT	1
HH	2

When we tabulate the number of heads, we see that there are two even cases and two odd cases so that the game is fair.

It's easy enough to look at three coins as well. Here there are 8 possibilities, 4 of which have an even number of heads. Again the game is fair.

Three coins	
outcomes	*# heads*
TTT	0
TTH	1
THT	1
THH	2
HTT	1
HTH	2
HHT	2
HHH	3

But the tables of possibilities are going to get big quite fast. We need to be more clever.

Now an interesting development—someone claims to have an argument that the game is always fair with an odd number of coins. We explore this result a bit, and in the end we find it convincing. In fact, two different ways of making the argument are produced.

Here's one. Associate to each outcome a "partner" obtained by replacing every H by T and every T by H. Then an outcome and its partner always have different parities—if one is even, the other is odd. That's because the total number of coins is odd. For example, in the three-coin table, the partner of any outcome is its mirror image in the line drawn halfway down the table. Since an outcome and it's partner have to be different (having different parities) the set of all outcomes is now partitioned as a collection of odd-even pairs, so there's the same number of odd outcomes as even, and the game is fair.

Three coins	
outcomes	*# heads*
TTT	0
TTH	1
THT	1
THH	2
HTT	1
HTH	2

Here's another. An outcome has an even number of heads if and only if it has an odd number of tails (and here's where we use the fact that the total number of coins is odd). So the probability of getting an even number of heads is the same as the probability of getting an odd number of tails. BUT, the situation with regard to heads and tails is entirely symmetric (reflect through the horizontal line half-way down the table at the right) so the probability of getting an odd number of tails has to be the same as the probability of getting an odd number of heads. We have shown:

Prob(even # H) = Prob(odd # T) = Prob(odd # H)

and the game must be fair.

So the game would be fair with 9 coins or with 11, but what about 10?
The argument does not seem to adapt to the case of an even number of coins.

But how are we to handle 10 coins? Someone suggests Pascal's triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Indeed, there are different ways of getting an even number of heads—0 heads, 2 heads, 4 heads, etc.—and the number of possibilities for each is given by the appropriate combinatorial coefficient. That is, to get 4 heads, we need to choose 4 of the coins to give heads, and the rest to give tails, and so the number of outcomes with that property will be $\binom{10}{4}$, the "4th" entry in the 10th row of Pascal's triangle which is 210. In fact, we can get a count of all the possible even numbers of heads by simply adding up the even-numbered entries in the 10th row of the triangle. We get:

$$\binom{10}{0}+\binom{10}{2}+\binom{10}{4}+\binom{10}{6}+\binom{10}{8}+\binom{10}{10} = 1 + 45 + 210 + 210 + 45 + 1 = 512$$

as the number of ways to get an even number of heads.

Now to make this into a probability, we need to know how many outcomes there are altogether? And the answer to this is 2^{10}—and that follows since there are two possibilities for each of 10 coins. You can of course check this by seeing that it is also equal to the sum of *all* the entries in the 10th row of the triangle. So the probability of getting an even number of heads is $512/2^{10}$ and that turns out to be exactly ½. Indeed, 512 is equal to 2^9 and that's half of 2^{10}, and the game is fair!

Well, we've solved the problem, but are we satisfied? *No!* What about other numbers of even coins? Is the game always fair for *any* number of coins? The only general argument we have so far works only for an odd number of coins. We can use the above triangle to check out the case of 4 coins, 6 coins and 8 coins—in every case the game is fair. Can we find a general argument for an even number of coins?

The first general argument we discovered works only for an odd number of coins. Finding one for an even number seems more difficult.

Look at the sum we generated for 10 coins:

$$1+45+210+210+45+1.$$

How might we *know* that these numbers added up to one-half of 2^{10}—without actually adding them up?

Well there's a beautifully simple argument for this using the additive structure of Pascal's triangle. The summands are the even-numbered entries of row 10. Now each of these (except the ends) is a sum of two row-nine entries. If we replace each by that sum, we get:

$$1 + 45 + 210 + 210 + 45 + 1 =$$
$$1+(9+36)+(84+126)+(126+84)+(36+9)+1$$

and that's just the row-nine sum!—and that's 2^9 which is half of 2^{10}.

This is a truly gorgeous argument. Why?—because it's so simple and direct and it uses the basic additive property that generates Pascal's triangle.

And of course it's completely general. It's works for any number of coins, even or odd.

This argument works in general— for all *n*, even or odd, and provides, in fact, an *inductive* proof that the game is always fair.

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Well, that all went much as I had figured it would, and I was pleased with how the class had gone. Until right at the end a student stood up and dropped a bomb. "What's wrong with *this* argument"? She began.

Are you ready for this?

Pick one of the coins and colour it red. Now toss the coins, look at the outcome, then pick up the red coin and turn it over. What we get is a different outcome *with the opposite parity*. That is, if the first outcome had an *even* number of heads, the second one has an *odd* number, and vice-versa. This device—changing the state of the red coin—gives us a simple way of partitioning the set of outcomes into pairs with opposite parities. It follows that there must be the same number outcomes with even and odd numbers of heads.

This argument is clear, simple, direct and completely general.

What I want to know is this: how is it possible to think about a problem for a few hours and never notice such a compelling argument? One of the great human mysteries I guess.

I was blown away.

Problems

1. Use Pascal's triangle to calculate the following probabilities:
(a) the probability of getting 5 heads in a toss of 7 coins.
(b) the probability of getting at most 4 heads in a toss of 9 coins.
(c) the probability of getting more heads than tails in a toss of 10 coins.

2. Eeyore and Owl play the following game. They roll ten dice, and Eeyore wins if the number of even outcomes (**2**, **4** or **6**) is even, and Owl wins if it's odd. Is the game fair, or does it favour one or the other?

3. I toss 10 nickels and 12 dimes and we note whether the number of heads of each type of coin is even or odd. I win if the parities are the same: both even or both odd, and you win if they are different, one odd and one even. Is the game fair?

4.(a) Which is more likely: exactly 5 heads in 10 coins or exactly 4 heads in 8 coins?
(b) Which is more likely: exactly 5 heads in 10 coins or exactly 5 heads in 9 coins?

5. You have to answer 10 true-false questions, and you haven't a clue on any of them, so you'll just have to guess.
(a) What's the probability of getting at least 5 correct?
(b) Now suppose you get 2 marks for each right answer and lose one mark for each wrong answer. To pass you need 10 marks. If you guess on them all, what's the probability of passing?
*(c) Same as (b) except now I want your <u>average</u> mark if you guess on all 10.

6. A colleague, Peter Liljedahl of Simon Fraser University, performed the following trick the other day. First he put on a blindfold. Then he asked me to take the coins out of my pocket, shake them up, dump them on the table, and tell him how many heads there were. There were in fact 8 coins, 3 of which were heads. He then said that, without taking off the blindfold, he would make two piles of coins with the property that the two piles had the same number of heads. That didn't seem possible. Was he going to feel with his fingers whether it's a head or a tail? *No, I don't need to do that, and I probably couldn't if I had to. But one thing I can do if I want is turn some of them over.* I thought about it a few moments and told him I couldn't see how he'd be able to do that without looking. He just smiled. I shook my head and told him to do it. He reached out with his hands, located the piles of coins, divided them into two piles, turned some of them over, and invited me to examine the result. He had indeed accomplished his objective, and there were in fact 2 heads in each pile. How did he do it?