Even odder

Eeyore and Owl play another game. They flip 10 coins, and Eeyore wins if the number of heads is even, and Owl wins if it's odd. Is the game fair, or does it favour one or the other? So far this is the same problem we had before (**Even odd**) where we discovered that the game was fair, but this time, the coins are *biased*. They each come up heads with probability 2/3 and tails with probability 1/3. Is the game still fair?

I originally thought that this might go well in the problem set of **Even odd** (and it would!), but I now see that it deserves a section of its own. There are two quite powerful but quite different approaches––one recursive and the other through Pascal's triangle.

I ask them to guess the answer, but their response is feeble. They shake their heads—their intuition fails.

So they decide to look again at the cases of a small number of coins, and they start with two. As before, there are four possibilities, but now they are not all equally likely, so we need a column for the probabilities. For example, to get HT you need a head on the first coin (probability 2/3) and a tail on the second (probability 1/3) for a net probability of 2/3 × 1/3 = 2/9. A good check is to note that the tabulated probabilities must add up to 1, and they do. Now the first and the last cases give an even number of heads, and their total probability is

$$P_2 = \frac{1}{9} + \frac{4}{9} = \frac{5}{9}$$

So in this case the game is not fair! but favours Eeyore.

Two coins		
outcomes	probability	# heads
TT	$\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$	0
TH	$\frac{1}{3}\times\frac{2}{3}=\frac{2}{9}$	1
HT	$\frac{2}{3}\times\frac{1}{3}=\frac{2}{9}$	1
HH	$\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$	2

Moving on to three coins, we now have 8 possibilities, but right away someone observes that we don't need 8 rows in the table—we can lump certain outcomes together. For example, in the two-coin table, we could have put together the two outcomes with one head and one tail as they each have the same probability. So, for example, with three coins, the three outcomes with 1 head and 2 tails, which are HTT, THT and TTH, all have probability $(2/3)\times(1/3)^2$ and can be lumped together. In this way we get a table with 4 rows, but for each we have to tabulate *the number of outcomes that are represented*, as well as the probability of each. I have reorganized the table in what seems a more natural way. And I have listed the outcomes as well as giving their number, so we can clearly "see" what is going on.

Three coins			
# heads	outcomes		probability
0	1	TTT	$\left(\frac{1}{3}\right)^3=\frac{1}{27}$
1	3	HTT THT TTH	$\frac{2}{3}\times\left(\frac{1}{3}\right)^2=\frac{2}{27}$
2	3	HHT HTH THH	$\left(\frac{2}{3}\right)^2\times\frac{1}{3}=\frac{4}{27}$
3	1	HHH	$\left(\frac{2}{3}\right)^3=\frac{8}{27}$

Now we are ready for the count. The total probability of an even number of heads (0 or 2) is

$$P_3 = \left(1\times\frac{1}{27}\right) + \left(3\times\frac{4}{27}\right) = \frac{13}{27}$$

and this is very slightly *less* than half. With three coins, the game favours Owl.

Again we check that the total probability is 1, where we have to multiply each probability in column four by the number of times it "occurs" which is found in column 2:

$$\left(1\times\frac{1}{27}\right)+\left(3\times\frac{2}{27}\right)+\left(3\times\frac{4}{27}\right)+\left(1\times\frac{8}{27}\right)=\frac{27}{27}=1$$

.

I wonder what the pattern is? We can give ourselves another data point by looking at the case of 1 coin—here the probability of an even number of heads (none) is clearly P_1 = 1/3, and the game also gives the advantage to Owl. We tabulate these three results at the right. Can we infer the 4-coin result?

# coins	Prob E wins	advantage
1	P_1 = 1/3	O
2	P_2 = 5/9	E
3	P_3 = 13/27	O

It doesn't take long to make a conjecture—the probabilities have denominators which are successive multiples of 3, and the numerators are alternately just below or just above half of the denominator. In particular, the "average" winner alternates—for even numbers of coins, the game favours Eeyore, and for odd numbers of coins, the game favours Owl. By this pattern, the 4-coin game should favour E with probability P_4 = 41/81 of winning. Let's check it out.

Four coins		
# heads	outcomes	probability
0	1	$\left(\frac{1}{3}\right)^4 = \frac{1}{81}$
1	4	$\left(\frac{1}{3}\right)^3 \times \left(\frac{2}{3}\right) = \frac{2}{81}$
2	6	$\left(\frac{1}{3}\right)^2 \times \left(\frac{2}{3}\right)^2 = \frac{4}{81}$
3	4	$\left(\frac{1}{3}\right) \times \left(\frac{2}{3}\right)^3 = \frac{8}{81}$
4	1	$\left(\frac{2}{3}\right)^4 = \frac{16}{81}$

We do this by constructing the 4-coin table. No doubt(!) you noted in the 3-coin table that the various numbers of outcomes are the third row of Pascal's triangle: 1 3 3 1. And here we have the 4th row: 1 4 6 4 1.

The total probability of an even number of heads is now:

$$P_4 = \left(1\times\frac{1}{81}\right) + \left(6\times\frac{4}{81}\right) + \left(1\times\frac{16}{81}\right) = \frac{41}{81}$$

as predicted!

So the prediction is looking good. Let's try to formulate it generally for the "n-coin" game. The denominator is 3^n and to get the numerator you either add (n even) or subtract (n odd) 1 from this and then cut the result in half:

$$P_n = \frac{\frac{3^n \pm 1}{2}}{3^n} = \frac{1 \pm 3^{-n}}{2}$$

# coins	Prob E wins	advantage
1	P_1 = 1/3	O
2	P_2 = 5/9	E
3	P_3 = 13/27	O
4	P_4 = 41/81	E

This lovely formula is predicted by the data in the above table. But we still have no general "proof" of its validity—at this point it is still a

So for the 10-coin game, the prediction is that it should very slightly favour Eeyore, with a winning probability of:

$$P_{10} = \frac{1+3^{-10}}{2} = 0.5000085 .$$

But is this the right answer? We could certainly check it out by actually constructing the 10-coin table, and doing the detailed calculation, but maybe that's not all that informative. What we *really* want is a general argument.

Is that nice simple P_n formula correct?
Is it valid for n=10?
For all n?

In working with this problem in the classroom, I have encountered two interesting arguments, one direct and one recursive. Both arguments exhibit important mathematical methods. The direct argument works with Pascal's triangle, as we did in **Even-odd**, and I discuss that in problem 10 below.

The recursive approach.
Suppose we knew P_9, the answer to the 9-coin problem. Could we find from there the answer to the 10-coin problem?

Well let's see. Suppose we flip 10 coins, 9 of them green and the other one red. [Note: we do not really know where we are going here or whether we will get there, but we know that we will want to involve P_9 in our argument so in colouring the coins in this way we are setting ourselves up for that.] We want to know when the total number of heads will be even. Now look at the green coins. Either there's an even number of heads or not, and the probabilities for each case are P_9 and $1-P_9$. Now in each case can we work out the probabilities for the total set of coins?

Yes we can. If there are an even number of green coins, then to stay even, the red coin better be tails (prob. 1/3), and if there are an odd number of green coins, then to get even, the red coin better be heads (prob. 2/3). So in a proportion P_9 of the cases, we get an even number of heads with probability 1/3, and in a proportion $1-P_9$ of the cases, we get an even number of heads with probability 2/3 . This gives us:

$$P_{10} = \frac{1}{3}P_9 + \frac{2}{3}(1-P_9).$$

Simplifying:

$$P_{10} = \frac{2}{3} - \frac{1}{3}P_9 .$$

This is the "recursive" formula we were after. It tells us how to find P_{10} in terms of P_9.

But does it allow us to actually find that formula for P_{10}? Well that's a question of *solving* the recursive equation. And it's not exactly clear at first how to go about that.

Whenever we have a family of problems indexed by the integers (for example, here we can consider the problem for 9 coins or 10 coins or 11 coins, etc.) it's natural ask whether we can move easily from one level to the next. If we happened to know the answer for the 9-coin problem, would that allow us to easily find the answer to the 10-coin problem? Such inductive or recursive ways of thinking can be very powerful.

This is called a linear first-order recursion, and there are standard ways to solve it. In fact we've seen one of these before in **Hitting 10**, and we will use the same "change of variable" method that we used there.

By the way, the recursion we met in **Trains** was also linear but it was <u>second</u>-order (the formula for t_{10} involved both t_9 and t_8) and a rather more sophisticated method is needed in that case.

In fact we've been here before, in **Hitting 10**. The idea is to take hold of the "limiting value" L of the sequence and then to work with the difference $x = P - L$. That's a "change of variable" which will simplify the recursion and allow us to solve it.

In this case the limiting value of P_n certainly appears to be $L=½$. That is, for large values of n, P_n approaches ½. So the variable to look at is the difference:

$$x_n = P_n - \frac{1}{2}.$$

Now what we have to do is write the recursion in terms of the x_n rather than the P_n. So we have to replace the P_n by x_n, and to do that we solve the above change-of-variable equation for P_n in terms of x_n:

$$P_n = x_n + \frac{1}{2}$$

and then substitute this into the P-recursion:

$$P_{n+1} = \frac{2}{3} - \frac{1}{3} P_n.$$

$$\left(x_{n+1} + \frac{1}{2}\right) = \frac{2}{3} - \frac{1}{3}\left(x_n + \frac{1}{2}\right)$$

$$x_{n+1} = -\frac{1}{3} x_n.$$

Now this can be solved. Starting with x_1 each x is obtained from the previous x by multiplication by $\left(-\frac{1}{3}\right)$ so that, for example, we get x_{10} with 9 such multiplications. In general:

$$x_n = \left(-\frac{1}{3}\right)^{n-1} x_1.$$

To find x_1 we recall that $P_1 = 1/3$, and we calculate

$$x_1 = P_1 - \frac{1}{2} = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}.$$

Then

$$x_n = \left(-\frac{1}{3}\right)^{n-1}\left(-\frac{1}{6}\right) \;=\; \frac{1}{2}\left(-\frac{1}{3}\right)^n$$

and hence

$$P_n = x_n + \frac{1}{2} = \frac{1}{2}\left(-\frac{1}{3}\right)^n + \frac{1}{2}$$

and this is exactly the conjectured P-formula.

The idea from **Hitting 10** is that there's a special case that's easy to solve, and that's the case when the recursion has no constant term:

$$x_{n+1} = kx_n$$

because this just says that x gets multiplied by k at each step. So if we know the starting value x_1 then x_n will be obtained by multiplying x_1 by k a total of n–1 times:

$$x_n = k^{n-1} x_1.$$

The important result is that we can change the above P-recursion into one of this type with the right change of variable.

Note in this last step all the constant terms have cancelled out. the $\frac{1}{2}$ on the left cancels with the $\frac{2}{3} - \frac{1}{3}\frac{1}{2}$ on the right.

Problems

1. Use Pascal's triangle to calculate the following probabilities:
(a) the probability of getting 3 sixes in a roll of 9 dice.
(b) the probability of getting 8 sixes in a roll of 9 dice.
(c) the probability of getting at most 1 six in a roll of 8 dice.
(d) the probability of getting an even number of sixes in a roll of 6 dice.

2. Eyeore and Owl play the following game. They roll ten dice, and Eyeore wins if the number of multiples of 3 (**3** or **6**) is even, and Owl wins if it's odd. Is the game fair, or does it favour one or the other?

3. (a) Suppose one play consists of flipping two fair coins. Player A wins if the coins have the same outcome, and player B wins if the coins have different outcomes. Is the game fair?
(b) Same as (a) except the coins are not fair—both coins come up heads 60% of the time.
(c) Same as (a) except the coins are not fair, one comes up heads 60% of the time, and the other comes up heads 40% of the time.
(d) Can you construct a fair game with biased coins?

4. Calculate
(a) the probability of getting an even number of sixes in a roll of 10 dice.
(b) the probability of getting six even numbers in a roll of 10 dice.

5. What's the least number of dice I have to throw for the probability of getting at least 2 sixes to exceed ½?

6. I throw 5 icosahedral (20-sided) dice with faces numbered from 1 to 20. What is the probability of getting at least three numbers which are divisible by 5?

7. (a) Suppose one play consists of drawing two balls from a bag containing 50 red and 50 green balls. Player A wins if the balls are the same colour, and player B wins if the balls are different colours. Show that the game is not fair.
(b) Can you make the game fair by recolouring the balls?

8. *A symmetric asymmetry.*
(a) I have 100 balls in an urn, coloured either black or white. If I draw two at random, the probability that they are of different colours is 50%. How many of each colour are there?
(b) In the above problem, replace 100 by N. For what values of N does the problem have a solution?

9. (*from Geoff Roulet*) I want to build a cottage this summer on the bank of a river, but I am told that the river floods every 20 years and that would surely sweep the cottage away. However, this spring was actually one of those bad years and the river has just finished flooding. In 15 years I plan to retire to the sunny south and I won't need that cottage anymore and I won't care two pins if it gets swept away. So I can build now and it'll stand for the next 19 years, so I'll certainly have it for as long as I need it.

What might be wrong with this argument? Suppose that successive years are treated as independent, and that 20 years is interpreted as the *average* time between successive floods. Show that my chances of having 15 flood-free years are less than 50%. [You need ideas from **Darts** here.]