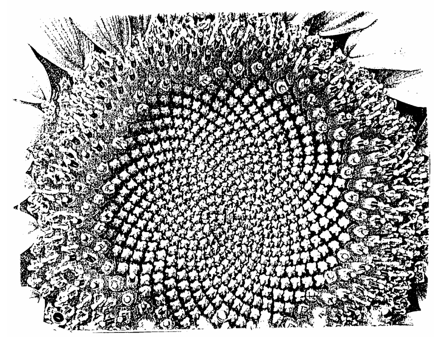
Sunflowers and pine cones

The seed pods on a cone are arranged in a spiral pattern: in fact there's a pair of spirals--if the cone is held vertically pointing to the sky, and we trace the spirals upwards, one goes around to the right (the *right* spiral) and the other goes around to the left (the *left* spiral). The spirals generally have different slopes too, one is steeper than the other, sometimes the right and sometimes the left.

Here's an amusement. Start at any pod, and trace up along both spirals until they meet around back of the cone. Count the number of steps taken along each spiral. You will always get a pair of consecutive Fibonacci numbers!! [Well, almost always—there are bound to be mutant forms lurking in the depths of the jungle, but they are rare, and in 10 years of sporadic gathering I've not encountered one yet.]

I name the cone according to the count. For example, a 3-5 cone is a cone which meets at the back after 3 steps along the *left* spiral and 5 steps along the *right*. For such a cone, the left spiral is the steeper of the two, and most of my small cones are 3-5, though I occasionally see a 5-3 (which meets at the back after 5 steps along the left spiral and 3 steps along the right). The bigger cones tend to be 5-8 or 8-5; I have examples of both kinds. I don't see many 8-13 cones—I have two of them (which look like they might be red pine) and they are getting old and dry and I caution the students to be careful with them.

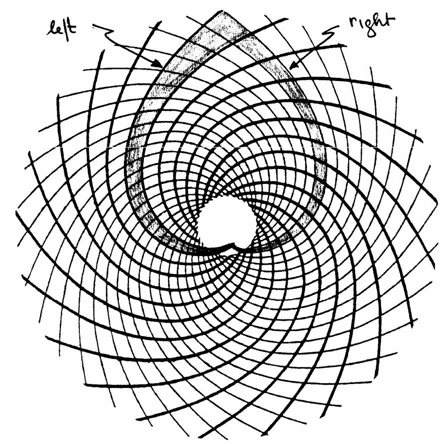


Other plants range their seeds in pairs of opposite spirals too. Pineapples tend to be 5-8 or 8-5 or 8-13 or 13-8. The most spectacular examples are found in the sunflowers. Here the seeds are tightly packed and one can get high numbers—at the right is a photo of a 34-21 sunflower, and there are also 55-34, 89-55, 144-89 sunflowers and their mirror images. I have an 89-55 which I bring to class. It's quite awesome to do the count.

Actually to compare the sunflower with the pine cone, you have to close the flower up and view the seeds from the bottom—then you'd be looking at a cone. What this means is that if you do look at the flower from above, what we've called the left spirals will run *clockwise,* and the right spirals will run *counterclockwise*.

Actually, the "cone" method of counting does not work well with sunflowers—if you tried to verify the count for the sunflower in the above picture, you probably ran into problems either at the periphery or the centre where there is disorganization and confusion. With a cone that is "opened out" like the sunflower, there is a much better way to do the counting. A geometric representation will make this clear.

Study the schematic drawing of the same sunflower at the right. A left (clockwise) and a right (counterclockwise) spiral have been shaded in. It's impossible to do the counting by running up these spirals—the resolution of the picture near the centre is simply not good enough. BUT there is a way that you can easily do the counting, right from the picture as given. Can you see how?



Well, take the shaded left (clockwise) spiral. We have to count the steps on that spiral. Now the shaded right spiral cuts this left spiral twice, once near the centre and once near the edge, and what we want to count is the number of steps between these two intersections. Now every such step is the intersection of the left spiral with another right spiral, *in fact there are as many steps as there are right spirals*. So we can count the steps by counting the number of right spirals, and this is easy enough to do, and we find that there are 34. The same argument works on the shaded right spiral—there are as many steps as there are left spirals, and we count 21 of these. So we have a 34-21 sunflower. So the sunflower type is calculated as R-L where these are the number if right and left spirals respectively. Use this idea to count the above photograph and check that it indeed 34-21.

Similarly, a 5-8 cone can be thought of as consisting of 5 distinct parallel spirals going right and 8 distinct parallel spirals going left.

So now that we've played a while with the pine cones and the sun flowers, it's time to ask the question that has certainly been burning to be asked:
Why do the Fibonacci numbers arise here?
Why aren't other pairs of numbers found?

One way to answer this question is on evolutionary grounds—that for some reason the Fibonacci numbers offer the best configuration for the plant in competing in the race for reproduction and survival. But what reasons might arise here? Competition for light or nutrients? Effective seed packing? When I throw this question out to the class, I can count on a variety of ideas, some inventive, some tempting, some crazy.

One set of ideas I always get revolves around the idea that the ratio of successive Fibonacci numbers is in some sense the best ratio "for the job." Indeed, the quotients 3/2, 5/3, 8/5, 13/8... approach the golden ratio τ and with a famous number like that your side, how could you go wrong?

The trouble with all arguments that draw on this ratio is that if I run the Fibonacci rule with another starting pair, say 1,3, the sequence we get:

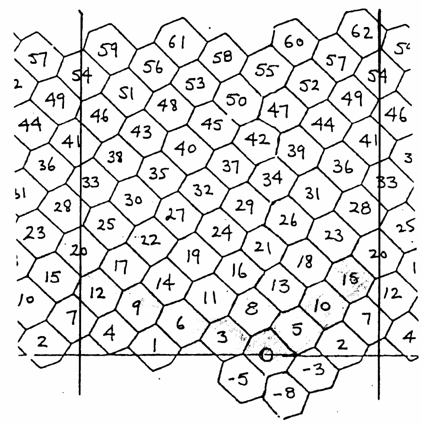
1 3 4 7 11 18 29 47 76 ...

also has the golden ratio property, that is, the ratios of successive terms also approach the golden ratio. So why don't we find any 4-7 or 11-7 pine cones, or 47-76 sunflowers? That's the real puzzle.

Here's another way to pose the dilemma. Consider an 8-5 cone which has just fallen off a tree in the park. That tree came from a seed in a cone—what kind of cone do you suppose it was?—no doubt 8-5 also. And the tree which that cone fell of—what kind of cone did it come from?—you guessed it: 8-5. Now keep going back in evolutionary time and we go through a long sequence of ancestral cones. I bet the earlier ones were smaller and simpler. I bet they used smaller numbers. In short, the 8-5 cone did not arise full-blown in some Adam-and-Eve fashion; surely it developed and grew out of more rudimentary cones under selective pressures to pack seeds more effectively. Let's suppose all the cones in this lineage had numbers. What would the set of numbers look like?

In fact, just to be specific, let's suppose that far enough back this 8-5 cone had a 5-3 ancestor. Now what might the intermediate forms look like? How does 5-3 turn into 8-5? Does it happen suddenly? Does it happen through a sequence of small jumps such as: 5-3, 6-3, 6-4, 7-4, 7-5, 8-5? If so, these intermediate forms were not Fibonacci—how long were they around and why aren't there any around today? Evolution is still going on—if non-Fibonacci numbers ever occurred, why don't we see any today?

Well, that's a very nice question. Amazingly enough, it has a simple answer which comes from studying the geometry of the spiral packing. Let's look closely at this supposed ancestor. In the diagram at the right, a 5-3 cone has been "unrolled" onto the page. As we move around the cone, we move sideways on the page, and eventually we return to where we started on the cone, but we have moved to a new place on the page.

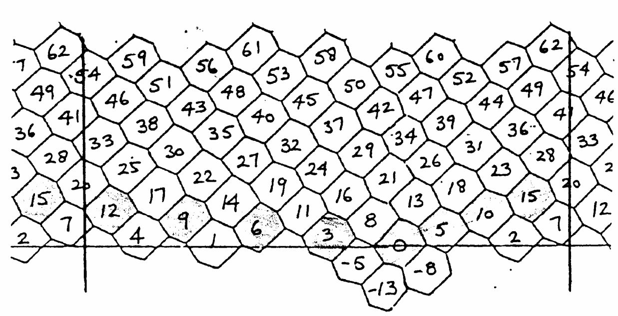


A fixed pod has been designated as 0, and all other pods are numbered by vertical height—thus 1 is just higher than 0, 2 just above 1, 3 just above 2, etc. As noted above, every pod appears once between the two vertical lines.

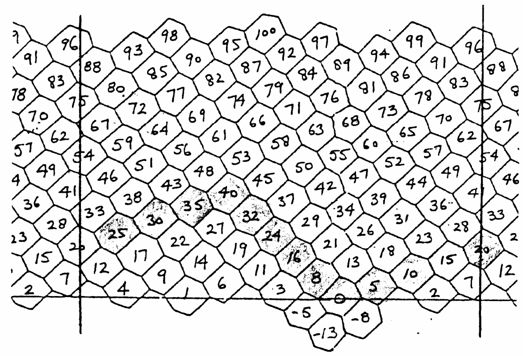
Now note how the numbers sit in the spirals. If we start at 0 and go up the left spiral, we go through the multiples of 3. If we go up the right spiral, we go through the multiples of 5. Of course we meet at pod 15 after 5 steps left and 3 steps right. Indeed, every left spiral goes up by 3's, and every right spiral goes up by 5's.

What other patterns can you see? Well one striking observation is that there is a third very steep spiral going to the left which counts every 8. If we go up this spiral, we will intersect the right spiral after 5 steps at pod 40, and to get there on that right spiral would require 8 steps. Perhaps you can see what's coming—we have a 5-8 cone waiting to happen.

How will it work? Under evolutionary pressure for a greater number of smaller seeds, the packing will compress down and get tighter and we will wind up with the spiral arrangement shown at the right. Note that the downward pressure has forced the 0,3,6,9... spiral apart, and the low spiral is now 0,5,10,15... going right, with the high spiral 0,8,16,24... (which used to be the very steep spiral) going left. And the new very steep spiral is 0,13,26,39... and it goes right. Along the way, the cone might grow a bit so I have expanded the horizontal scale.

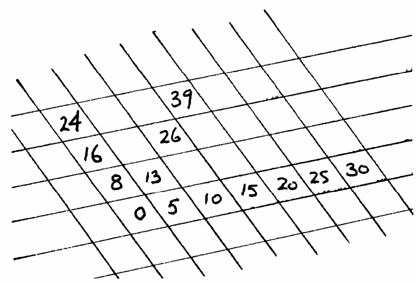


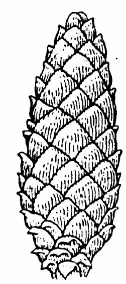
Of course, the point is that we'll get more seeds on the cone (for a fixed height), so that new seeds will come on line and the whole picture might be more like that drawn at the right. Whereas the 3-5 cone had about 60 seeds, the 8-5 cone might have about 100. Interestingly enough, 60 and 100 are in the ratio 3:5.



Be sure to note that this cone is not an 8-5 but a 5-8! The natural progression is for a 5-3 to evolve into a 5-8, and similarly, a 3-5 cone will evolve into an 8-5.

The amazing thing is that we don't need evolutionary "jumps" at all—the process can be accomplished in a completely gradual manner. The 5-3 cone will stay a 5-3, but the slope of the spirals will gradually come down, and then one day it will start to look more like a 5-8 than a 5-3.





What will the transitional form look like? Well pod 3 will be just about to separate from pod 0 and pod 13 will be just about to connect to pod 0 from above––so pod zero will have two real neighbours, 5 and 8, and two casual acquaintances, 3 and 13. Thus the hexagonal packing looks, for a brief evolutionary moment, more rhombic than hexagonal.

There's one more problem. What we've shown is that once you have a Fibonacci cone, its evolutionary descendants will also be Fibonacci, perhaps using higher pairs of numbers. But why did things start in a Fibonacci way? If an early cone had been 2-5, it would have given rise to a 7-5 descendent, followed by 7-12, etc. Why didn't this get going?

Well, what are the likely starting places? There's the possibility of 1-1 and 1-2 and these are both Fibonacci. The first possibility for a non-Fibonacci sequence is a 1-3 start, and then comes 1-4, etc., and these seems to me (and perhaps natural history bears me out) far less likely than 1-1 or 1-2. Anyway, that's the best I can do for now.

Problems.

1. As an explanation for the ubiquity of the Fibonacci numbers, I have had my students suggest that perhaps the spiral packings drawn above are only possible for consecutive Fibonacci numbers. But this is not the case. For example, use the template at the right to construct a 4-7 cone (which might evolve from a 1-3 start).