Fibonacci Numbers

This is certainly the most famous "sequence" in mathematics—but it is still an object of continued study by mathematicians. In fact, it has spawned a professional journal, The Fibonacci Quarterly.

The Fibonacci sequence $\{u_n\}$ starts with 0 and 1, and then each term is obtained as the sum of the previous two:

$$u_n = u_{n-1} + u_{n-2}$$

The first fifty terms are tabulated at the right.

u_0	0
u_1	1
u_2	1
u_3	2
u_4	3
u_5	5
u_6	8
u_7	13
u_8	21
u_9	34
u_{10}	55
u_{11}	89
u_{12}	144
u_{13}	233
u_{14}	377
u_{15}	610
u_{16}	987
u_{17}	1597
u_{18}	2584
u_{19}	4181
u_{20}	6765
u_{21}	10946
u_{22}	17711
u_{23}	28657
u_{24}	46368
u_{25}	75025
u_{26}	121393
u_{27}	196418
u_{28}	317811
u_{29}	514229
u_{30}	832040
u_{31}	1346269
u_{32}	2178309
u_{33}	3524578
u_{34}	5702887
u_{35}	9227465
u_{36}	14930352
u_{37}	24157817
u_{38}	39088169
u_{39}	63245986
u_{40}	102334155
u_{41}	165580141
u_{42}	267914296
u_{43}	433494437
u_{44}	701408733
u_{45}	1134903170
u_{46}	1836311903
u_{47}	2971215073
u_{48}	4807526976
u_{49}	7778742049
u_{50}	12586269025

Our objective here is to find arithmetic patterns in the numbers—an excellent activity for small group work. I simply hand the students the list of the first 50 numbers, and tell them to *Go nuts!*

They put their proposals up on the board. Sometimes these are variations of something I already know, typically a version of some standard relationships like those given below, and sometimes they blow me right out of the water.

Of course there's lots of patterns having to do with squares. Such as:

$$3^2 + 5^2 = 34$$
$$5^2 + 8^2 = 89$$

and

$$5^2 - 2^2 = 21$$
$$8^2 - 3^2 = 55$$

and

$$8^2 - 5^2 = 39 = 3\times 13$$
$$13^2 - 8^2 = 105 = 5\times 21$$

And here's one with products:

$$2\times 5 + 3\times 8 = 34$$
$$3\times 8 + 5\times 13 = 89$$

These become quite satisfying when we use the u_n notation, because we can see how the subscripts relate. Here are the formulae above in *u*-notation. Note particularly the subscript patterns.

$$u_4^2 + u_5^2 = u_9$$
$$u_5^2 + u_6^2 = u_{11}$$

and

$$u_5^2 - u_3^2 = u_8$$
$$u_6^2 - u_4^2 = u_{10}$$

and

$$u_6^2 - u_5^2 = u_4\times u_7$$
$$u_7^2 - u_6^2 = u_5\times u_8$$

$$u_3\times u_5 + u_4\times u_6 = u_9$$
$$u_4\times u_6 + u_5\times u_7 = u_{11}$$

And there are lots more. See the problem set.

Mathematical Induction.

Mathematical induction provides one of the standard ways to establish formulae like those presented above. It can work particularly naturally for Fibonacci number properties as the numbers themselves are generated recursively. Sometimes the inductive argument is straightforward, and sometimes it's not and requires some ingenuity, and that's always fun. Let me start with a simple example.

*Occasionally a "physical" model of the Fibonacci numbers (see **Trains**) or just the right geometric picture (problem 5 below), can give us an unexpected argument for some of these Fibonacci identities. Those are my favorite kinds of proofs. But here we will look briefly at the inductive approach.*

Example 1.

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 = 8\times 13.$$

Check it out—both sides are equal to 104 and the equation holds. That's certainly a nice property. Just to run a check, verify that the "next" case also holds:

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 = 13\times 21.$$

Now how did you check this new equation out? Did you add up the squares all over again? In fact, all you had to do was add 13^2 to what you had the first time and check that was equal to 13×21. And that insight contains the key to the inductive method.

The idea behind mathematical induction is to show that the truth of the first equation just *has to* imply the truth of the second. If we can show this in a "general" way, so that it can be seen to apply not only for this pair of equations, but for any such pair, then the argument can be applied repeatedly to show that the whole infinite set of equations must hold.

Let's try it. We assume the first is true and we want to establish the second. To do that we'd better write down the sum on the left:

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2.$$

Our target is to show that this equals 13×21, but what we do *not* want to do is simply calculate it—we want to use the truth of the first equation. Well, the obvious thing to do is to substitute the first part of the sum:

$$(1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2) + 13^2$$

$$= (8\times 13) + 13^2.$$

$$= 8\times 13 + 13\times 13.$$

$$= (8 + 13)\times 13$$

$$= 21\times 13$$

and that's exactly what we want.

What we do here is to manipulate the expression with the target in mind. It's usually not quite clear what to do though—we "feel" our way forward. In this case, we have 13 in both terms and it seems reasonable to pull it out as a common factor.

Okay. Are all equations like this true? Just to go a bit farther down in the list, are you now convinced that:

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + \ldots + 233^2 = 233^2\times 377^2 \text{ ?}$$

Have we a "proof" that this is true? Does the above argument convince us that all future instances of the formula will hold?

I ask the class whether, without doing the calculation, they are convinced that this equation holds. The vote is mixed. Perhaps some of them worry that my standards of proof will be higher than theirs.

Well, let me start by saying that it certainly convinces me. Having seen the above argument, I am quite sure that I could make it work at any stage, with any length of sum. But of course the reason I am so convinced is that the argument has been set down in a careful way, in a way that allows me to see the general in the particular. The numbers that appear in the argument can always be identified as particular Fibonacci numbers, so that in the last step when 8+13 is replaced by 21, that is seen to use the Fibonacci property and we know that it will always work like that.

A conventional mathematical "proof," would ensure the generality of the argument by using the symbols u_n and u_{n+1} instead of 8 and 13. But my own preference is for the argument with numbers instead of the symbols u_n. In that rather more concrete form, I find it easier to understand and work with—for example, if I need one day to extend it.

Interestingly enough if one mathematician walked into the office of another wanting to "prove" that the above relationship held, the argument written on the board would more likely be the specific one with numbers that appears above than the general one with symbols that appears below.

Having said that, I'm going to end with the formal argument. In order to "see" that an argument with particular numbers really is general, you need to know what a general argument looks like!

Proposition. $1^2 + 1^2 + 2^2 + 3^2 + 5^2 + ... + u_n{}^2 = u_n u_{n+1}$

Proof: By induction. We first observe that the equation holds for the case n=1:

$$u_1{}^2 = u_1 u_2$$

since both u_1 and u_2 are equal to 1. Now *assume* the truth of :

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + ... + u_n{}^2 = u_n u_{n+1}.$$

We need to show

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + ... + u_n{}^2 + u_{n+1}{}^2 = u_{n+1} u_{n+2}.$$

Take the left side:

$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + ... + u_n{}^2 + u_{n+1}{}^2$

= $u_n u_{n+1} + u_{n+1}{}^2$ [Induction hypothesis]

= $(u_n + u_{n+1}) u_{n+1}$ [Common factor]

= $(u_{n+2}) u_{n+1}$ [The Fibonacci property]

and that's exactly what we needed to show.

Mathematical Induction. *You have a sequence of statements or equations and you want to show that they're all true. You argue that if any one is true, then that implies that the next one must be true. If you can do that, and you can show that the first one is true, then you know that the second, and then the third, and then the fourth, etc. all must be true. So they're all true.*

Example 2.
Use induction to establish the "sum of squares" pattern:

$$3^2 + 5^2 = 34$$
$$5^2 + 8^2 = 89$$
$$8^2 + 13^2 = 233$$
etc.

Solution. One of the notable things about this pattern is that on the right side it only captures *half* of the Fibonacci numbers. Subtract the first two equations given above:

$$\begin{array}{r} 5^2 + 8^2 = 89 \\ \underline{3^2 + 5^2 = 34} \\ 2\times 8 + 3\times 13 = 55 \end{array}$$

The numbers in the bottom equation have been produced by the difference of squares formula. For example:

$$5^2 - 3^2 = (5-3)(5+3) = 2\times 8.$$

Now if we somehow "knew" that the bottom equation was valid, we'd have a proof that the "34" formula implies the "89" formula, because we'd just add the bottom equation to the "34" formula to get the "89" formula.

An interesting thing about the bottom equation is that everything that appears is a Fibonacci number. So the bottom equation itself represents a Fibonacci identity. For example, the "next" in that sequence would be:

$$3\times 13 + 5\times 21 = 144.$$

So now we have two families of Fibonacci identities and we have seen that if we could establish the second family, we'd have an inductive argument for the first. Well maybe it works the other way 'round too: if we knew the first family we'd have an inductive argument for the second. In fact that turns out to be the case!

So where does that leave us? *Catch 22?*—if we know the first we get the second and if we know the second we get the first. What can we do with that?

Plenty. We can mount a *double* induction and carry both families along together!

That's the idea. Can you fly with it? If you get stuck, turn the page.

*We see a spectacularly simple and elegant argument for these identities in **Trains**. In view of that, I thought it would be nice to include an inductive argument in this section, so I sat down to try to fashion one. And I tried and I tried and got very frustrated. Holy cow. It looks simple enough—what could the problem be?*

The solution I finally stumbled upon is the one I report here. It's really quite wonderful and it also illustrates an important mathematical method.

Before you read this solution, be sure to play with the problem a bit and try different inductive approaches. I'm sure there are others that work.

This is interesting. We have two families of identities:

$$2^2 + 3^2 = 13$$
$$3^2 + 5^2 = 34$$
$$5^2 + 8^2 = 89$$

and

$$1\times 5 + 2\times 8 = 21$$
$$2\times 8 + 3\times 13 = 55$$
$$3\times 13 + 5\times 21 = 144$$

Each family uses only half of the Fibonacci numbers, but they use opposite halves!

It turns out that each family holds the key to the inductive proof for the other. That sets us up for a double inductive process.

The double induction.

We take a pair of adjacent equations, one from each family and assume that they hold. Let's take:

$$3^2 + 5^2 = 34$$
$$2\times 8 + 3\times 13 = 55$$

Now we want to use this to show that the next pair must also hold:

$$5^2 + 8^2 = 89$$
$$3\times 13 + 5\times 21 = 144$$

Well the first is essentially done—it's exactly what we observed at the beginning. Look again at the difference calculation on the previous page. It tells us that if we add the first two equations (the "34" and the "55") we get the third (the "89").

So it remains to establish the fourth equation (the "144"). Why not add the second and the third and see if we get the fourth?
Well certainly we get the correct right side (144). So let's look at the sum of the left sides:

$$(2\times 8 + 3\times 13) + (5^2 + 8^2)$$

and we want to show that this is equal to $3\times 13 + 5\times 21$. Well we can certainly cancel the 3×13 terms to get the equation:

$$2\times 8 + 5^2 + 8^2 = 5\times 21.$$

We have to show that this is true. Remember—we don't want to do this by simply calculating both sides and seeing that we get the same answer on both sides. We *know* that would happen. To get a general solution, we want to see how the Fibonacci pieces of the equation come together to make it true.

Well let's try to show that there are 21 fives on the left side. Writing it as

$$2\times 8 + 5\times 5 + 8\times 8$$

we see that we have 5 fives and a total of (2+8) eights. Now

$$(2+8) = (2+3+5) = (5+5)$$

so that's the same as (5+5) eights, and that's the same as (8+8) fives. So the total number of fives is:

$$5+(8+8) = (5+8)+8 = 13 + 8 = 21$$

just what we wanted to show.

Here are the two families:

$$1^2 + 1^2 = 2$$
$$1^2 + 2^2 = 5$$
$$2^2 + 3^2 = 13$$
$$3^2 + 5^2 = 34$$
$$5^2 + 8^2 = 89$$

and

$$0\times 1 + 1\times 3 = 3$$
$$1\times 3 + 1\times 5 = 8$$
$$1\times 5 + 2\times 8 = 21$$
$$2\times 8 + 3\times 13 = 55$$
$$3\times 13 + 5\times 21 = 144$$

The inductive idea is to assume that a pair of equations holds, one from each family, and prove that the next pair must then hold.

For this example, it's a nice challenge to construct the formal general argument from all of this. Care and focus are needed and I leave that task to you. [Problem 2]

Problems

1. What I often do, after I give the class the table of the first 50 Fibonacci numbers to pour over, is to say: who can tell me the sum of all the Fibonacci numbers on the page: $u_0+u_1+u_2+...+u_{50}$? I like this challenge because it recalls the famous question that Gauss was given at age 12 when he was asked to sum the first 100 natural numbers. In that case, the teacher was hoping to keep the class quiet for a hour, but Gauss came up with the answer in a few moments. I certainly do not expect anyone to add up all 51 numbers. What is it I am hoping they will do?

2. Use the calculations of Example 2 to provide a general proof by induction (using the u_n notation) for the sum of squares formulae:

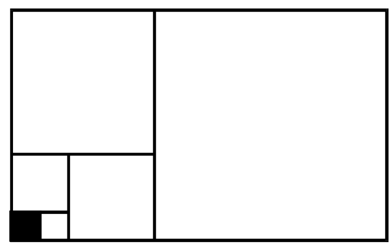
$$u_n^{\ 2} + u_{n+1}^{\ 2} = u_{2n+1} .$$

3. The difference of squares formula, of which an example is $u_7^2 - u_6^2 = u_5 \times u_8$ is easy to prove directly. The left hand side is dying to be factored.

5. The formula from Example 1:

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 = 8\times 13.$$

has a particularly nice geometric proof . Use the picture at the right to find it. Does this proof generalize to other formulae in the family?

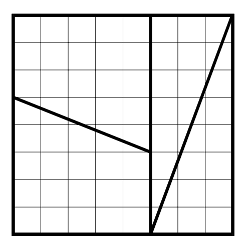


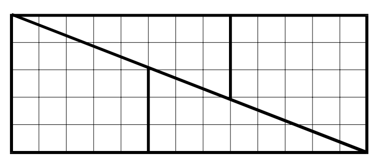
8. Is the relationship $\begin{Bmatrix} 34-1 \\ 55 \\ 89-1 \end{Bmatrix} = 11\times \begin{Bmatrix} 3 \\ 5 \\ 8 \end{Bmatrix}$ part of a general pattern? [Hint: try to construct a similar relationship with 2,3,5 in the right hand bracket instead of 3,5,8.]

13. Consider that:

$$8^2 = 5\times 13 - 1$$
$$13^2 = 8\times 21 + 1$$
$$21^2 = 13\times 34 - 1$$
$$34^2 = 21\times 55 + 1$$

(a) Use mathematical induction to show that this pattern continues.
(b) These identities give rise to a famous geometrical paradox illustrated by the diagram at the right for the case $8^2 = 5\times 13 - 1$. The rectangle and the square are composed of the same 4 pieces, yet the rectangle has area 65 and the square has area 64. Go figure.





21. Take the Fibonacci sequence 1 1 2 3 5 8 13 etc. and divide the first term by 100, the second by 1000, the third by 10000, etc. and then add them all up (to infinity)—the sum is 1/89. Wow.
[Hint: The standard approach to finding the sum of an infinite geometric series is to multiply the series by something (r) and then subtract the two versions of the series, one from the other, and see what we get. The same type of trick might work here.]

34. Verify that $(3\times13)^2 + (2\times5\times8)^2 = 89^2$. Is this part of a general relationship? This is amusing because it gives us a family of Pythagorean triangles.

55. $1\times1 + 1\times2 + 2\times3 + 3\times5 + 5\times8 + 8\times13 = ?$
Play with this one. Write down quite a few of the sums of this form. Play with them. It may take a while to see even what pattern there might be, but the pattern is quite stunning and is a nice candidate for a double induction. [Hint: why did I say “double” induction? Look for a different pattern for odd and even terms.]

89. $$\frac{1}{1\times 2}+\frac{1}{1\times 3}+\frac{1}{2\times 5}+\frac{1}{3\times 8}+\frac{1}{5\times 13}+\frac{1}{8\times 13}+\ldots=1$$
Can you establish this?

144. Take any row of Pascal's triangle. Multiply the nth entry of the row by u_n , and add everything up. What do you get? [You get a formula of Cesaro.] For example, for the fifth row:

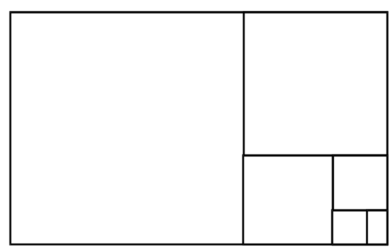
$$1\times1 + 5\times1 + 10\times2 + 10\times3 + 5\times5 + 1\times8 = ?$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

233. Tabulate the cubes of the first few Fibonacci numbers. Try to find patterns by adding and subtracting. There’s a nice one to be found, but I suspect it’s not easy to establish inductively or otherwise.

377. Show that the quotients, 1/1, 2/1, 3/2, 5/3, 8/5, ... of successive Fibonacci numbers approach the golden ratio τ defined as the positive root of the equation $x^2 = x+1$.
[Find a recursive relationship for the quotients $r_n = u_n / u_{n-1}$].

610. The Greeks had the idea that the rectangle of the most pleasing proportions is the one with the property that if you cut out a square, what you are left with has the same shape as before. That means we can keep going removing squares forever, and never lose the shape. Show that the two sides of this rectangle are in ratio τ, the golden mean.



987. (A problem of Steinhaus) Consider the sequence beginning with $x = 1$ and $x = a$ which satisfies the recursion:

$$x_n = x_{n-2} - x_{n-1} \ .$$

Find all values of a for which the sequence has only positive terms. [Hint: solve #12 first, and look carefully at how the sequence in #12 approaches its limit.]