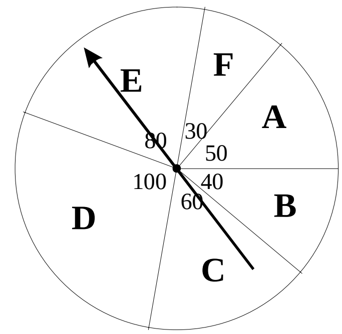
First hit probability



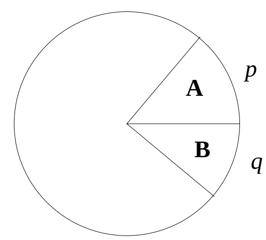
For the spinner illustrated at the right, the pointer is spun at a high rate and stops at a random place. There are six pie-shaped regions of different angular widths, and the probability that the pointer will stop in any one is proportional to its angle. Thus the probability the pointer will hit A is 5/36 and the probability the pointer will hit B is 4/36. Here's the problem:

Find the probability that the pointer will hit A before it hits B.

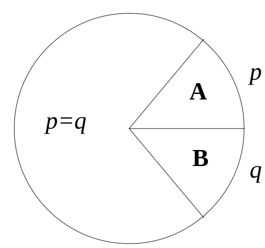
That is, if we spin the pointer again and again and stop only when either A or B is hit, what is the probability that A will get hit first?

Your intuition can take you quite a way with this problem. Since A is bigger than B it seems reasonable that it will have a greater chance of being hit first. But how much greater? One intuitive idea is that since the A and B areas are in the ratio of 5 to 4, their probabilities of being hit first should also be in the ratio of 5 to 4. In that case, what does that make the probability that A will be hit first?

It might take a moment to work that out, but a good way to say it is that out of 9 times that we do the experiment, we expect A to be hit first 5 times and B to be hit first 4 times. So the probability that A is hit first should be 5/9. That actually turns out to be the right answer, but it's not se easy to find a convincing argument.



The general problem is this. If in any spin, A has probability p of being hit and B has probability q of being hit, what, in terms of p and q, is the probability that A will be hit first?

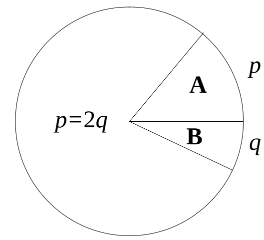


A good strategy is to look for simple cases that you can find good arguments for. One obvious one is the case $p=q$ where A and B are the same size. In that case, how could one of the two have a greater probability of being hit first than the other? This is a simple symmetry argument—they have to have the equal probability of being hit first, so each probability must be ½. Thus:

$$P(A) = \text{prob. A is hit first} = \tfrac{1}{2}\,.$$

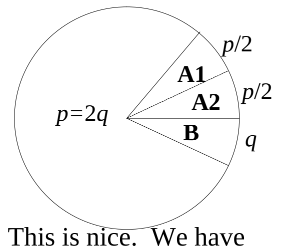
That's a start. Where do we go from here?

Now let's not move too fast. Take another case, almost as simple. Suppose that A is twice the size of B, so that $p = 2q$. What should P(A) be then?



One feels that in this case, P(A) should be twice as big as P(B). But how can we make that precise? Is there any way to use the simple symmetry idea we found above?

Yes there is. Cut A in half. That gives us three regions of the same size which we will call A1, A2 and B. Now ask for the probability each of these will be hit first (before the other two) and by symmetry this has to be the same for each one, and is thus 1/3. So,



$$P(A1) = P(A2) = P(B) = 1/3.$$

Now A will be hit before B precisely when A1 or A2 are hit first. Thus:

$$P(A) = P(A1) + P(A2) = 2/3.$$

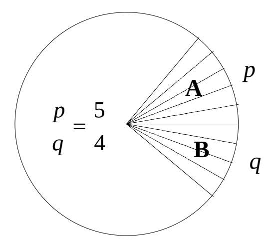
This is nice. We have found a way to make that elegant symmetry argument work when p=2q.

How powerful is that symmetry argument? Can we make it go "all the way"? Let's try the problem we started with in which

$$p = 5/36$$
$$q = 4/36$$

Will the same type of construction work?

Indeed it will. To make the symmetry argument work, what we need are regions of equal size. We can arrange that by dividing A into 5 equal pieces and B into 4 equal pieces. Then we have 9 sectors all of the same size and by symmetry the probability any one of them will be hit before the others is always the same and therefore must equal 1/9. It follows that A will be hit first with probability 5/9.



It should be now clear that this same argument will work for any p and q which are in the ratio of two integers, that is, whose ratio is rational. How do we write that 5/9 answer in terms of the p and the q?

It's $\dfrac{p}{p+q}$. The general result is this.

> *First-hit probability.* In any trial, let the outcome A occur with probability p and the outcome B occur with probability q. Then in a sequence of trials, the probability that A occurs before B is
>
> $$P(A \text{ before } B) = \frac{p}{p+q}.$$

Our proof of this by symmetry only works when the ratio p/q is rational, but since any real number is a limit of rationals (take the successive truncations of its infinite decimal expansion) a limiting argument can be used to establish the result for any p and q.

There is however an algebraic proof which calculates the probability directly. It does this by breaking the event "A before B" into sub-events, each of which can be easily handled. Here's how it goes.

In any class I've done this problem with there is always a student who wants to work out this "analytic" argument.

Each time A occurs before B it will happen on a certain trial, maybe the first, maybe the second, etc. This infinite sequence of possibilities is tabulated below. Beside each of these, the probability of that sub-event is given. The overall probability that A is hit before B is the sum of these sub-event probabilities. We use $r = 1-p-q$ to denote the probability that the outcome of a spin is neither A nor B.

sub-event	**probability**
A on the 1st trial.	p
neither A nor B on the first trial; A on the 2nd.	rp
neither A nor B on the first 2 trials; A on the 3rd.	r^2p
neither A nor B on the first 3 trials; A on the 4th.	r^3p
neither A nor B on the first 4 trials; A on the 5th.	r^4p
etc.	

The events listed in the table are *disjoint*, that is, at most one can occur on any trial. In this case, the probability that one *will* occur is the sum of the individual probabilities (and that's the probability that A is hit before B).

The overall probability of A being hit before B is the sum:

$$\begin{aligned} \text{P(A before B)} &= p + rp + r^2p + r^3p + r^4p + \dots \\ &= p[1 + r + r^2 + r^3 + r^4 + \dots] \\ &= p\frac{1}{1-r} = \frac{p}{p+q} \end{aligned}$$

and this establishes the result.

Sum of an infinite geometric series

$$a + ar + ar^2 + ar^3 + \quad = \frac{a}{1-r}$$

I must say, I also like to work out the analytic argument, just to see that it gives the same answer as the symmetry argument. But for me the argument from symmetry is the proof of choice, not so much because it's simpler (though it is!) but because it follows my intuition; it truly convinces me; it's how I see *why* the result ought to hold.

Problems

1. A single die is loaded according to the table at the right. If it is rolled a number of times, find the probability that:
(a) a 6 appears before a 5.
(b) a 6 appears before any odd number.

A	P(A)
1	0.1
2	0.1
3	0.1
4	0.2
5	0.3
6	0.2

2. Suppose I roll a fair die again and again. Calculate the probability that:
(a) I roll at least one 1 before I roll a 2.
(b) I roll at least one 1 before I roll any even number.
(c) I roll at least two 1's before I roll a 2.
(d) I roll at least one 1 *and* at least one 2 before I roll a 3.

3. Suppose I roll a pair of fair dice again and again. Each roll, my *score* is the sum of the outcomes and my *total score* is the sum of all my scores so far. What is the probability that my total score will be at least 8 *before* I roll a 1 (on either die)?

4. The discussion of this section has been in the context of trials which are *independent*. By this I mean that in a series of trials, the outcome of a single trial has no effect on the probabilities of different outcomes of subsequent trials. This is the case for a spinner or for the roll a die. The probabilities don't change in successive trials (unless for example spinning the spinning increases the friction or rolling the die blunts some of the corners etc.) However this is not the case for dealing successive cards from the top of a shuffled deck. For a standard deck (4 suits, 13 cards in each) the probability that the first card dealt is an ace is 4/52 = 1/13. What is the probability that the second card dealt is an ace? Well there are two answers for that. First of all, the probability that the second card dealt from a well-shuffled deck is an ace is 1/13. [Indeed *any* card in such a deck is an ace with probability 1/13.] But secondly, *if we know the first card*, then that information changes the probability that the second card is an ace. The situation is as follows:

- if the first card is not an ace, the probability the second card is an ace is 4/51.
- if the first card is an ace, the probability the second card is an ace is 3/51.

This problem assumes that the student understands the discussion so far.

(a) Take a well-shuffled standard deck and deal cards from the top until either an ace or a face card (J, Q or K) appears. What is the probability that an ace will appear first? The main result of this section would take p=1/13 and q=3/13 and would deduce that an ace would occur first with probability ¼. Does this result hold in this case? Can you use the symmetry argument to establish it?
(b) Suppose I have 5 cards, 2 red, 2 white and 1 blue. If I shuffle them and deal them from the top, calculate the probability that the blue card will appear before either of the red cards. Do this in two ways:
(i) using a symmetry argument
(ii) constructing a table of the different possible ways this could happen: on the first card, on the second card, etc.