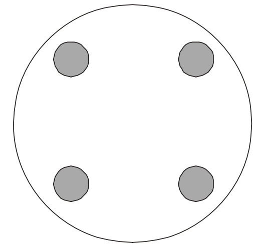
Four Wells

A circular table has 4 deep wells, equally spaced around the perimeter, and in each well there is a glass, oriented either up or down. You cannot see the glasses, but if you reach into any well with your hand you can feel the orientation of the glass. Here's the game: you close your eyes and the table is rotated into a random position. You can then stick your hands into any two of the wells, discover the orientation of the glasses, and then if you wish, reverse one or both of the glasses. Then the game repeats again and again: you close your eyes, the table rotates, etc. If at any stage all four glasses have the same orientation, a bell rings and you have "won." The problem is, can you choose a strategy which will guarantee a win in a finite number of moves?



Some time ago, this clever problem was all the rage in the mathematical community and it was written up by Martin Gardiner in Scientific American 240 no. 2 and 3 (1979). The problem is a tad sophisticated, and I was wary about using it with high school students, but it turned out to work exceedingly well. It takes a bit of time to get onto the problem, but once into it, it has the flavour of a mystery story with different intermediate configurations arising which have to be resolved. Different groups had fun presenting their arguments, and along the way, they were introduced to an important mode of mathematical thinking.

The nice thing about the problem is that there are easier variants to start with—like having fewer wells, or even, more hands. Except sometimes that might make the problem harder!—we'll just have to see. For example, 3-wells are easier than 4, and on a 4-well table, 3-hands are easier than 2. In fact we can consider the general game with *n* wells and *m* hands, and I find the students keen to "collect" the cases which they can solve.

In fact, I suggest they start with the case of 3 wells and 2 hands, and before long Katharine is up at the board to show how to ring the bell. Note that the glasses cannot all be the same at the beginning or the game would be over. In the first round, make the two glasses the same, say AA. Then either the bell will ring or it won't. If it doesn't, the third glass must be B. On the next probe, you get either AA or AB. In the first case, invert both glasses, and in the second invert B.

To keep track of our progress, we constructed a matrix of all the (m, n) possibilities, and put a W in the slot if we had found a procedure which would win in a finite number of bouts, and an L if we could show that no procedure could exist. The above argument allows us to enter a W in the (3-well, 2-hand) slot, and of course we can also put W's in all the slots directly underneath.

	# wells n				
# hands m	2	3	4	5	6
1					
2		W			
3		W			
4		W			
5		W			

To completely settle the 3-well story, we have to decide whether 1 hand is enough to win. If you think about it a moment, it certainly seems possible that you might never manage to ring the bell, but a careful argument is needed, and the presentation of this is a good exercise for the students. Again you must have an AAB configuration or the bell would have rung. If you put your (single) hand in one of the A wells, then whatever you do, you will still not ring the bell. And so again you will have the same type of configuration AAB or ABB. So if you happen to *always* put your hand in that duplicated well, you'll never ring the bell. We can insert an L at the start of the 3-well column.

	# wells n				
# hands m	2	3	4	5	6
1		L			
2		W			
3		W			
4		W			
5		W			

A digression

Last time I worked with this problem an interesting and highly instructive encounter occurred at this point. Sumit had just presented the above L argument for 3 wells and 1 hand, when Mario raised his hand.

I disagree. I think I can see how to win. The answer should be W.

Okay, Mario, you're on.

Well, I always turn the glass over, no matter whether it's up or down.

Okay.

Then there'll always be a 1/3 chance that I'll ring the bell—that's because if the bell hasn't rung, I know I have an AAB, and there's 1/3 of a chance I'll have chosen the B.

I asked the class if they agreed, and they did. No problem so far.

So as long as the bell doesn't ring, there's always 2/3 of a chance that I won't *ring the bell.*

Yep.

Well as I get more and more bouts, the chances that I'll never ring the bell get smaller and smaller. And approach zero. So sooner or later I'll ring the bell. For sure.

There was a silence.

Go ahead—set it up—I guarantee you I will ring the bell. He smiled happily.

I was beginning to see the sense in which Mario was right and I saw that I hadn't stated the problem carefully enough. Anyway, it was a tricky point, somewhat technical, but very important. To buy a few minutes of time, I threw the dilemma out to the class. And in the discussion which followed, Mario gained a number of supporters.

This player-adversary formulation is completely clear and it seems to me to sharpen and enable our thinking processes as well. So henceforth, this is the statement of the problem we will work with. I might have started this section simply with that statement, but I felt that the Mario episode was worth including.

Well, it *is* a tricky point, and worth coming to grips with. The problem does have to be restated in a more precise manner and there are a number of ways to do that. Here is the nicest of these I have encountered. Imagine the problem as a two-person game between a *player*, who can't see the glasses, and wants to ring the bell, and an *adversary*, who can see the glasses, and who will do everything in his power to thwart the player. Then we suppose that at each bout, the player must first choose the position of his hands above the table, and then the adversary *who knows the state of all the glasses,* can rotate the table to any position he chooses. And then the player drops his hands into the wells which are now positioned under them. With this interpretation, it is clear that Mario, as the player, will never ring the bell, as the adversary will always rotate an AAB table so that Mario's hand is over an A.

So this "clarification" of the problem is the one that we will now assume and work with.

Back to the problem!

There are other cases which are quite easy to settle, for example a winning procedure is easily found for the (4-well, 3-hand) problem, and the (5-well, 4-hand) problem, and the (6-well, 5-hand) problem. In fact, see problem 1 below. Also it is not hard to show that 2 hands are not enough for both 5 wells and 6 wells (see problem 2).

	# wells n				
# hands m	2	3	4	5	6
1	W	L	L	L	L
2	W	W		L	L
3	W	W	W		
4	W	W	W	W	
5	W	W	W	W	W

The rest are a bit trickier, and as examples, I will do the (5-well, 3-hand) analysis (NO) and the (6-well, 4-hand) analysis (YES).

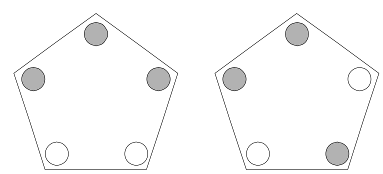
5 wells—3 hands not enough

If the glasses are not all the same, then there are two things that must be true:

1) there must be two *adjacent* glasses that are different.
2) There must be two *non-adjacent* glasses that are different.

Condition 2) takes a moment's thought. Take any glass, and then go around the table clockwise testing every *second* glass. After four moves you've tested every glass (this uses the fact that 5 is odd) and since they're not all the same, at some point you must have got different results on two successive tests. That's two non-adjacent glasses that are different.

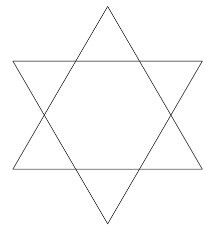
Okay, now suppose I am the player and you are the adversary. I claim that no matter how I put my 3 hands, you can rotate the table so that there are two different glasses that are not under me. Indeed, there are really only two ways I can put my hands, three together, or two together and one separate. In the first case, you use 1) and in the second case, you use 2).

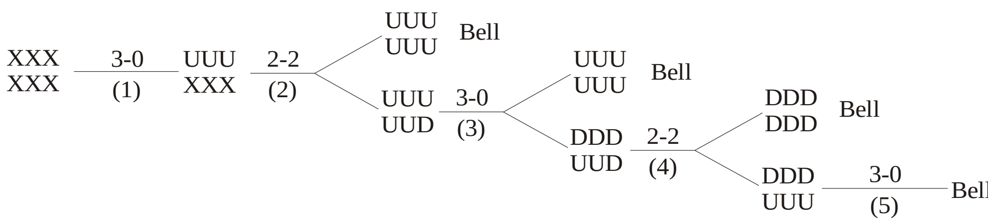


6 wells—4 hands will suffice

The algorithm the class came up with for this was rather complicated and unrolled as sequence of "cases." I thought about it a bit, and I remember feeling that there just might be a simpler approach but couldn't really see how to do it. So I gave the write-up to my graduate student Andrew to think about over the weekend (after all—what are graduate students for?), and Monday morning he presented me with the solution I reproduce below—a substantial improvement, not only on the algorithm itself, but on its presentation. He uses a tree diagram instead of an enumeration of cases, and employs a more compact notational scheme than we had before. That's always an important question in the presentation of a solution—how much detail to carry along?—too much will get in the way of the flow, but too little will lose the reader. Andrew's comments were interesting. He said that it took him "a while to warm up to the problem," but once he got into it he found it quite neat. His work emphasized for me the two rather different aspects of this problem—first of all, finding and "honing" a solution, and secondly, which is just as important, deciding on the "right" way to present it. Both are good challenges for your students.

In the tree diagram below, U means "up," D means "down," and X means "unknown.," To represent the placement of the hands, we will think of the six wells as composed of two equilateral triangles, and we consider two types of placements—one which samples two vertices from each triangle, and we represent that as 2-2, and the other which uses only three hands and samples all three vertices from a single triangle, and we represent that as 3-0.





One must study each step carefully, to ensure that the transitions can in fact be made. For example, after (2), if we do not hear the bell, we *know* we must have UUU/UUD. Note that after (1), it is possible that the bell will ring, but we have omitted to note that.

To finish the table off, similar arguments show that 3 hands are not enough for 6 wells, and that 2 hands *are* enough for 4 wells (that's the one we started with) and I leave them for the problem set (problems 3 and 4).

	# wells n				
# hands m	2	3	4	5	6
1	W	L	L	L	L
2	W	W	W	L	L
3	W	W	W	L	L
4	W	W	W	W	W
5	W	W	W	W	W

***n* wells, *m* hands**

The general (n-well, m-hand) game has been analyzed and solved by Ted Lewis and Stephen Willard, and they obtain an elegant but quite unexpected condition! Their result is that a winning strategy exists precisely when

$$m \geq \frac{p-1}{p} n$$

where p is the largest prime divisor of n. For example, the largest prime divisor of n=6 is p=3, and the condition says that a winning procedure will exist when there are at least $\frac{2}{3}6 = 4$ hands.

When I first saw the Lewis-Willard result, it astounded me. Given the complexity of the various cases, and the different kinds of arguments that seemed to arise, a simple general condition seemed hopeless. But here it is, and a curious one it is too!

How are we to interpret it? Here's a start: for those numbers n whose largest prime divisor is 3, there have to be 2/3 as many hands as wells. For those numbers n whose largest prime divisor is 5, there have to be 4/5 as many hands as wells. Etc. Why does the largest prime divisor arise here? You can get some feeling for that by trying to generalize the (5-well, 3-hands) argument. See problem 6.

Problems

1. Argue that you can always win with n wells and n-1 hands.

2. Show that 2 hands are not enough for the 5-well game.

3. Show that 3 hands are not enough for the 6-well game. [Hint. Cast yourself in the role of the adversary. Think of the hexagon as made up of two disjoint equilateral triangles. Note that if each of these triangles is uniform (all A or all B), then you, as the adversary have lost—that is, the player can win by making an equilateral triangle of his hands. So your job is to keep this from happening. Show that if one of these triangles is mixed, you can always rotate the table to keep it mixed no matter how the player positions his hands.]

4. Show that 2 hands are enough for the 4-well game.

5. If k hands are not enough to win the n-well game, does it follow that k hands are also not enough to win the (n+1)-well game? Use the above Lewis-Willard formula to answer this.

6. In the case that n is prime, p and n are the same in the Lewis-Willard formula and it tells us that the player needs at least n–1 hands to win, that is, n–2 are not enough. The (5-well, 3-hand) argument is a special case of this, but you really need to take a bigger value of n to see what's really happening.
(a) Show that 9 hands are not enough for a win at the 11-well table.
(b) Show that if n is prime, then n–2 hands are not enough for a win at the n-well table.

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