Handshake

A standard combinatorics problem supposes there are n people in a room, and everyone shakes hands with everyone else. The problem is: how many handshakes are there in all?

A high school teacher called me up the other day and said they had done the problem in his Finite Math class that day, and that one of the students had come up at the end and asked if he knew how long it would take to perform all those hand-shakes. He asked me if I had encountered the problem before. I hadn't and neither had any of my colleagues. That's surprising as it turns out to be rather nice.

I thought about it at odd moments over the next few days, and was surprised that I didn't see right away what to do. In fact, there were a number of approaches that seemed to suggest themselves, and I wasn't sure which to follow. Even though it was a busy time for me, the problem kept "repeating" like a spicy onion. Well, I had a high school session coming up in a couple of days, so I decided to stop thinking about the problem, pose it for the group, and just see what happened. It turned out rather well.

We have to pin it down a bit. What we really have here is a nice MIN problem--what is the minimum time required to perform all the handshakes if you can only shake hands with one person at a time? Actually, I find when my students start to think about minimum time, some of them worry about people bumping into one another, and finding their new partner etc., so a good way to pose the problem is to suppose that hand-shaking is organized into bouts and you can only shake hands once every bout, and the bouts are spaced out enough for everyone to find their designated partner. Then the problem is, what's the least number of bouts required for everybody to shake with everybody else? And what would a "schedule" of partnering look like which attained the minimum?

I started the session by posing the original problem: what's the total number of handshakes? Some students already knew the answer (and a couple even had a formula they'd memorized) so what I asked is that they find as nice a way as possible to present or "understand" the answer.

The number of handshakes.

Even students in grade 9 can handle this if they start with small concrete values of n and go slowly and carefully. For cases like n=3 or 4, you can list all the handshakes and count them--maybe you can see a pattern in the numbers. Better still, try to list them in an organized way, and look for a pattern in the "shape" of the list, which tells you what the general answer should be. For example, for n=4, we have the following list

12			
13	**23**		
14	**24**	**34**	
15	**25**	**35**	**45**

The first column is all of the 1-shakes, the second is all of the 2-shakes that haven't been counted before, etc. Many students found this array, and concluded that the number of handshakes with 5 people is:

$$H(5) = 4+3+2+1 = 10$$

and in general, with n people,

$$H(n) = n+(n-1)+\ldots+2+1 = ?$$

What's the sum of the series? Isn't there a formula for that? Some remembered the formula, or guessed at it, and a couple of older students even remembered the algebraic trick used to find it. But the neatest way is to use the above shape we already have for n = 5—it's half of a rectangle formed by taking a copy of it, turning it upside down, and fitting the two pieces together:

12	**45**	**35**	**25**	**15**
13	**23**	**34**	**24**	**14**
14	**24**	**34**	**23**	**13**
15	**25**	**35**	**45**	**12**

The double rectangle is $4 \times 5 = 20$, and the answer is half that which is 10. In general, the whole rectangle has size $(n-1)n$, and the answer is half that which is

$$H(n) = \frac{(n-1)n}{2}.$$

Incidentally, our argument for the sum of the series is actually a geometric version of the standard algebraic argument.

But the best solution of all was yet to come. First count the number of hands that get shaken. Since each of the n people shakes hands n–1 times, there are $n(n-1)$ instances of a hand getting shaken. Since every handshake involves 2 hands, there must be half that many handshakes. Simple and very powerful.

A rather neat argument

In fact, that last inspired approach provides an unusual argument to find the formula for the sum of the first n numbers:

$$1+2+3+\ldots+(n-1)+n = \frac{n(n+1)}{2}.$$

Here's how it goes. Imagine there are n+1 people in a room and everyone shakes hands with everyone else. Count the number of handshakes in two different ways. Using the first method above (n shakes for A, then n–1 remaining for B, then n–2 for C, etc.), we get the sum on the left, and using the second method (count the number of hands shaken and divide by 2) we get the expression on the right, so they must be equal.

The minimum number of bouts

Now we proceed to the problem of finding the minimum number of bouts. The students started by working out schedules for small numbers. They quickly established $T(2)=1$, $T(3)=3$, $T(4)=3$, and $T(5)=5$, where I use $T(n)$ to denote the minimum time required.

For most of the session there was a fair bit of diversity in the room—different students with different ideas—but it was nicely creative, and by putting suitable items of progress up on the board, we stayed fairly focused.

A general observation was that there was a difference between odd n and even n, that for odd n there'd be some unavoidable "waste of time" since at every bout of shaking, at least one person would be left out. Whereas for even n, it's possible (at least theoretically) for everyone to be shaking in every bout.

In fact this led us to the question of lower bounds. For even n, the best possible result would be obtained if everyone participated in every bout. Since each person has n–1 hands to shake, this would give a time of n–1 minutes. So we know that for even n,

Proposition 1. The even lower bound.

For n even, $T(n) \geq n-1$.

This lower bound is evidently attained for n=2 and 4. Is it attained for other n?—for every even n?

point that we had showed that the min time for even n was n–1, and we had solved the even case. But they weren't thinking clearly. What we've shown is that n–1 is a lower bound on the

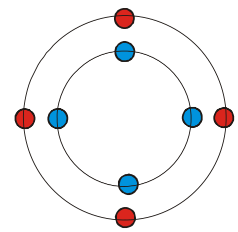
What's the corresponding lower bound for odd n? In this case, every bout will leave someone out, so n-1 bouts will not be enough, and we need at least n. If there was no additional waste, so that every bout involved n-1 individuals (and $(n–1)/2$ shakes), then n minutes would do it, since we'd have a total of $\frac{n(n-1)}{2}$ shakes in n bouts, and that's just the number we have to get. Thus, our odd lower bound is

Proposition 2. The odd lower bound.

$$\text{For } n \text{ odd, } T(n) \geq n .$$

Again, we have yet to show (or even know!) that this lower bound is always attained.

Some new results came in—first that $T(8)=7$, so the lower bound is attained here. The argument decomposes the room into two groups of 4, the "red" group and the "blue" group. First we do all the "within-colour" shakes—reds shake with reds, and blues with blues. That takes 3 bouts (since $T(4)=3$). To get the "between-colour" shakes, put the reds in a circle, and the blues in another circle, inside the red circle, and then rotate the blue circle 90 degrees after each bout. We'll get all the pairs in 4 bouts. Essentially, what we've done is use the n=4 result to get the n=8 result, with the formula:



$$T(8) = T(4)+4 = 3+4 = 7$$

This argument can be continued, and we get:

$$T(16) = T(8)+8 = 7+8 = 15$$

In this way we can establish the result for all powers of 2:

Proposition 3. The lower bound is attained for powers of 2.

$$\text{For } n = 2^k\text{: } T(n) = n–1.$$

The next result that was announced was

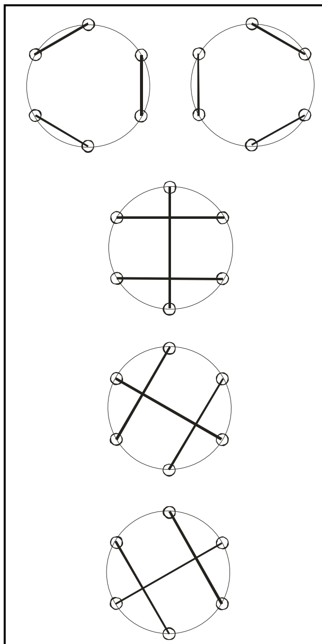
$$T(6) = 6$$

and thus 6 did not meet the even lower bound of n–1. The announcer used the above decomposition, dividing the six individuals into two groups of three, performing the 3 within-group shakes first, and then having the 3 between-group shakes. In terms of the formula, we have:

$$T(6) = T(3)+3 = 6.$$

But no sooner was this done, than someone claimed to have constructed a schedule for 6 people with 5 bouts. Impossible! said the announcer, and pointed to his decomposition formula as a "proof" of his result. But of course, this formula is not a proof, but only an observation that a particular method of constructing a schedule won't give the lower bound.

Indeed, the lower bound of n–1 can be attained for n = 6, and a schedule is diagrammed at the right.



***T*(6) = 5**

This was an interesting "counterexample" as it pointed out the need for care in making assertions that something couldn't be done

A number of small results/conjectures were reported, but the next big breakthrough came from some efforts to relate the odd and even configurations and results were produced that went both ways. Specifically, if n is odd (and therefore $n+1$ is even) and we have the lower bound for one, we also have it for the other.

Proposition 4. An odd-even relationship. For every odd-even pair ($2n$–1, $2n$) the lower bound is attained for one if and only if it is attained for the other:

$$T(2n-1) = 2n-1 \quad \text{if and only if} \quad T(2n) = 2n-1.$$

For example, this tells us that if we could show that $T(11) = 11$ then we could deduce that $T(12) = 11$. And if we could show that $T(12) = 11$ then we could deduce that $T(11) = 11$.

We'll do the proof for the (11, 12) pair. Suppose we know that $T(11) = 11$. Take a room with 12 people and ask one of them to sit down. Now and make a schedule for the remaining 11 with 11 bouts. In each bout there will be exactly one person left out—there couldn't be more than one as we need a total of 55 shakes, and that means 5 shakes per bout. Furthermore, that person will be different each time, as everyone needs 10 shakes. Okay, all we have to do is have the seated person shake hands each time with the person left out and the 11 bouts will serve the 12 people.

To go the other way, take a room of 11 people and add a "dummy" to make 12 and construct a schedule for the 12 people with 11 bouts. Well, that gives an 11-bout schedule for the 11 people with the person who shakes the dummy hand being the one who is left out.

This leaves us in an interesting position: if we can establish one of the lower bounds in general, odd or even, then we also have the other one.

Making induction work.

One group of students had gone off on a quest to make the inductive procedure which was used for the powers-of-two case, work for any even number—and with a little checking and reorganization here and there, they were successful. Recall the procedure—since there are an even number of people we can divide them in half—red hats and blue hats. Then we use the inductive hypothesis to perform the within-colour shakes and then rotate the circles to get the between-colour shakes.

Proposition 5. The lower bound is attained for every even number. For every even number n, $T(n) = n$–1.

In fact there are two cases, depending on whether the number of hats of each colour is odd or even.

Proof by induction. The details are best presented with a particular number, though the argument is quite general. We will illustrate the two cases with the numbers 12 and 14. Assume that the result holds for every even number less than 12.

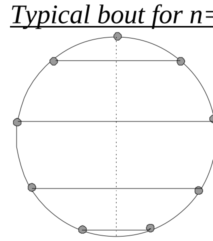
Take a room with 12 people and put red hats on 6 of them and blue hats on the other 6. By the inductive hypothesis, the within-colour shakes which involve 6 people will take $T(6)=5$ bouts. Finally it is easy to get all the between-colour shakes in 6 bouts—form two concentric circles, one with the red hats and one with the blue. Then rotate the red circle 6 times so that each red hats gets to meet each blue hat. In total, the number of bouts is 5 + 6 = 11, and that's the lower bound for 12.

Now take a room with 14 people and put red hats on 7 and blue hats on the other 7. By the inductive hypothesis, the within-colour shakes will take $T(7)=7$ bouts (since 7 is odd), and, using the two circles as above, the between-colour shakes will also take 7 bouts for a total of 14 shakes. But the lower bound for 14 people is 13 bouts, so we somehow have to remove a bout. Think about the first round of 7within-colour shakes. Observe that in each of these there will always be one person left out—indeed one person of each colour—and in fact everyone will be left out exactly once, as everyone has to shake hands 6 times. So in that first round, we get those two people to shake hands. Then when all the within-colour shakes have been done, everyone will have performed exactly one between-colour shake. So when we form the two concentric circles, the red circle and the blue circle, put each person opposite the person of the other colour that he has already shaken hands with. The rotate one of the circles 6 times to get the remaining 6 shakes for each person. This gives us a total of 7+6=13 bouts . This is the lower bound for 14 people so we conclude $T(14)=13$.

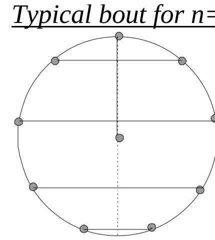
Proposition 5 gives us the lower bound for all even *n*. We deduce from Proposition 4 that we have the odd lower bound too.

It would seem that we have now solved our problem, and everyone ought to be able to go happily home. Nevertheless, the most spectacular result of the afternoon was yet to come and it was one I had not "found" on my own! I had been aware of a student sitting in the back who was working with a regular *n*-gon, trying to find an elegant scheduling algorithm based on symmetry. I had considered this approach myself for $n=6$, trying to use symmetries of the 6-gon, but hadn't been able to see what to do. So anyway, I was starting to wrap the session up, when this student raises his hand and asks "what's wrong with this?" and he comes up to the board and draws a simple construction for $n=9$. Place the 9 people at the vertices of a regular 9-gon. Pick a vertex and draw a line through the centre passing through that vertex. Now get everyone to shake hands with their mirror image in that line, where the person intersected by the mirror is the one left out. We do this with each vertex in turn, and get 9 bouts with everyone left out once. So simple. I was impressed.

Typical bout for n=9



Typical bout for n=10



Does it work in general? Well, yes it does—for odd *n*. What about even *n*? Well that took us a moment to see. The trick is to work not with the *n*-gon but with the $(n–1)$-gon, and put the last person at the centre. Then again we use a mirror through each vertex, except this time the person on the mirror always shakes with the person in the centre. That gives us $n–1$ bouts.

Well, this last minute "discovery" dismayed me somewhat. Suddenly the solution to the problem was real easy—just a simple construction, and none of the arguments we had developed before this point were at all necessary. But of course there's good thinking and learning in them all. And isn't it so often the case that the simple elegant solution only comes at the end?

Problems

1. Calculate the following sums:

(a) $1+2+3+\ldots+100$
(b) $2+4+6+\ldots+100$
(c) $1+3+5+\ldots+99$
(d) $5+10+15+\ldots+100$
(e) $55+60+65+\ldots+100$
(f) $57+62+67+\ldots+102$

One inevitable topic in the senior math curriculum is the sum of an arithmetic series. The formula that the student "learns" is soon forgotten, and problems like 1(f) become inaccessible. But the sequence of problems on the left shows the student how the answer to 1(a) allows us to arrive at the answer to 1(f), and this process is much easier to "reinvent" at a future time. Of course, this is also the process by which the general formula is arrived at.

2. Suppose that *n* married couples arrive at a party and everyone shakes hands with everyone else, except of course that there are no handshakes between spouses. How many handshakes are there in all?

3. Suppose that 4 married couples arrive at a party and that there are a number of handshakes, but none between spouses. At some point, each of the 8 people are asked how many handshakes they made, and it is discovered that all 7 possible answers (the numbers from 0 to 6) are obtained at least once. What was the total number of handshakes?

4. Suppose there are *n* people in the room, and everyone must shake hands with everyone else. How many "bouts" are required, if each person is allowed to shake hands with two others at once (using both hands). Note that a handshake must be "proper" in that both parties must be using the right, or both the left. For example, if there were 3 people, we'd need 2 bouts, and 4 could also be done with 2 bouts, but 5 people would require 3 bouts.