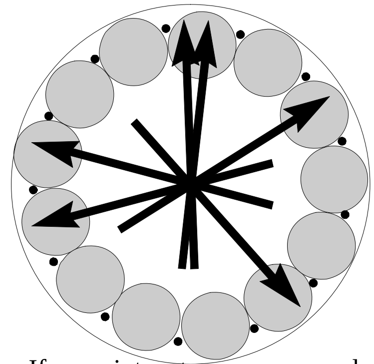
Horse Racing



Danny and I were at the "Ex" the other evening, and we came across a game of chance known (for reasons that are not clear to me) as horse racing. There are 13 pads arranged around the outer rim of a circle, and a stack of 6 pointers which are spun, each pointer stopping at a random pad. Furthermore, the pointers seem to be of different weights, so they rotate at different speeds and can presumably be treated as independent. You put your dollar on any pad, and then your payoff is determined by the number n of pointers which stop at your pad. If $n=0$, you simply lose your dollar. But if $n \geq 1$, you get your dollar back plus n dollars.

If no pointer stops on your pad you lose your dollar. If 1 pointer stops on your pad you get your dollar back plus 1 dollar. If 2 pointers stops on your pad you get your dollar back plus 2 dollars. Etc.

Danny played the game 8 times, and wound up 6 dollars ahead. *Of course, I was just lucky*, he explained, *the game actually favours the House*. How do you see that? *Well, it would be fair if there were twelve pads, so 13 pads gives the advantage to the House*. I asked him to explain his twelve-pad result.

Uh, okay. If there are 12 pads, then each pointer has probability 1/12 of stopping at your pad, and so, on average, you get 6/12 = 1/2 dollar as payoff each play of the game, and you also get your own dollar back with probability 50%, giving you another half dollar, and those two half dollars just balance the dollar you put down.

Hmm. Not a bad argument to produce "on the fly," but I decided it needed some careful thought. Do you agree?

Solution.
I suggest to the class that, rather than work with the original 13-pad game, we focus on Danny's claim that the 12-pad game is fair.

I begin by giving them five minutes in their groups to talk about the problem. At the end of that time the first comment I get is from one of my best students.

" Well you've done it again––given us a problem which I think I ought to be able to do but seems just one tiny tad too tricky to handle."

In fact this is not really such a difficult game to analyze, but the payoff situation does at first seem complicated and one has to see how to think about it

So what shall we do?—*construct a simpler problem!* Of course. *Fewer pads!* Right on.

How few can we get? Well, an essential feature would appear to be that there are twice as many pads as pointers. So the "smallest" game would have two pads and one pointer. So let's examine the "2-pad" game.

The 2-pad 1-pointer game. There are only two possibilities—you either get the pointer or you don't—and they are equally likely. And we can easily tabulate the payoff in each case. If you don't get hit, you lose your dollar for a payoff of –1, and if you do, you actually come out 1 dollar ahead. The (weighted) average of these payoffs is

$$P_2 = \frac{1}{2}(-1) + \frac{1}{2}(1) = 0$$

and the game is fair—the average payoff is zero!

2 pads		
# hits	*prob.*	*payoff*
0	1/2	–1
1	1/2	1

Well, I wonder if the game is always fair. I take a vote and the class is split. I think most of them would have bet on it being fair at this point, except that I *did* suggest at the beginning that I was skeptical of Danny's argument. But then I could have had a subversive reason for saying that…

The 4-pad 2-pointer game. Again we tabulate the three possible outcomes and the payoff for each case. We have to work a bit to get the probabilities. For example to get the probability of 1 hit we need to choose the pointer that makes the hit, and there are 2 ways to do that, and then that pointer will hit the right pad with probability ¼ and the other will miss the same pad with probability ¾. When we calculate the (weighted) average, we get

$$P_4 = \left(\frac{3}{4}\right)^2[-1] + 2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)[1] + \left(\frac{1}{4}\right)^2[2] = \frac{1}{4^2}[-9+6+2] = -\frac{1}{16} = -0.0625$$

4 pads		
# hits	*Prob.*	*payoff*
0	$\left(\frac{3}{4}\right)^2$	–1
1	$2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)$	1
2	$\left(\frac{1}{4}\right)^2$	2

Aha!—the game is no longer fair. The average payoff is slightly negative. Well, that's interesting. Let's do the next case just to see if this holds up. And besides, we want to start to try to see if we can understand what's really happening.

The 6-pad 3-pointer game. This time there are four possible outcomes. As an example of the calculation, to get the probability of 2 hits we need to choose the two pointers that makes the hit, and there are 3 ways to do that, and then those two pointers will both hit the right pad with probability $(1/6)^2$ and the remaining pointer will miss the same pad with probability 5/6. The weighted average is

$$P_6 = \left(\frac{5}{6}\right)^3[-1] + 3\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)[1] + 3\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^2[2] + \left(\frac{1}{6}\right)^3[3]$$

$$= \frac{1}{6^3}[-125+75+30+3] = -\frac{17}{216} = -0.0787$$

6 pads		
# hits	*prob.*	*payoff*
0	$\left(\frac{5}{6}\right)^3$	–1
1	$3\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)$	1
2	$3\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^2$	2
3	$\left(\frac{1}{6}\right)^3$	3

Again this favours the house, and by a slightly greater margin than the 4-pad game.

So it looks as if the 12-pad 6-pointer game might well favour the house. Interesting.

Well, what should we do?
We could construct the table for the 12-pad game and get the answer? Yup. We sure could. Who wants to do it? No response. Well, that's okay—I don't much want to do it either. I mean I'm pretty well convinced it's going to favour the house. Just what the exact numerical probability is—well… who cares?

This is one of those times when the natural tendency of students to avoid putting up their hands can be turned to advantage.

But there *is* something we *are* interested in: *the reason it favours the house, the flaw in Danny's reasoning.* Yes, indeed. We still yearn to get a better understanding of the problem. Right?!

I fix the class with a beady eye––Anyone feel that "yearn" is too strong a word?
Not a sound.

The problem is that the payoff rule is a tad complicated making it hard to think clearly about it. The answer is to simplify the rule and one way to do that is to decompose the payoff as a sum of two simple pieces. I find it takes the class a while to arrive at this, and they typically follow an indirect route. One approach, for example, starts by studying the 6-pad table and noticing that the first and last columns are almost the same. So look at their difference…

Analysis of the game. There are two parts to the payoff and it helps to imagine that they are separately administered by two people, a woman who pays you the number of hits on your pad and a man who takes away your dollar when there is no hit on your pad. Letting the payoff from the woman and man be A and B, then your total payoff P is:

This is a key step and illustrates a basic mathematical method. When faced with a complex quantity, analyze it by breaking it up as a sum of simpler components.

$$P = A + B$$

A = # hits on your pad
B = 0 if there is a pointer, –1 if there is no pointer.

In any of the tables above, each row has an A and a B. For example, the A and the B are depicted for the 6-pad game. Each row also has a probability and what we want in each case is the *average* value of A and B taken over all rows. And because each of these has a simple interpretation, the two averages turn out to be easy to find. The average payoff P is then obtained by adding these two averages up.

	6 pads		
# hits	*prob.*	*A*	*B*
0	$\left(\frac{5}{6}\right)^3$	0	–1
1	$3\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)$	1	0
2	$3\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^2$	2	0
3	$\left(\frac{1}{6}\right)^3$	3	0

Indeed the average of A is the average number of hits on your pad, and since there are half as many pointers as pads, this will be ½.

$$\text{Avge(A)} = \text{avge \# hits} = ½$$

The average of B is "minus" the proportion of times there is no hit. Let's work that out for the 6-pad 3-pointer game. Take a fixed pad. Now each pointer will miss that pad with probability 5/6, so the probability all 3 pointers will miss the pad is the cube of this: $(5/6)^3$.

$$\text{Avge(B)} = – \text{probability of no hit} = (5/6)^3$$

Putting it all together:

$$\begin{aligned} \text{Avge(P)} &= \text{Avge(A)} + \text{Avge(B)} \\ &= ½ – (5/6)^3 \\ &= 0.5 – 0.5787 \\ &= –0.0787 \end{aligned}$$

just what we got before.

This is the same numerical answer we got before, but this is a much more powerful derivation. We have it analyzed into two pieces in a way that can be easily generalized to any game.

We will no longer need the table with all its rows.

The 12-pad 6-pointer game. We can now write down the answer for the 12-pad game almost immediately. First, with 12 pads and 6 pointers, the average number of hits on a fixed pad is still ½:

$$\text{Avge(A)} = \text{avge \# hits} = \tfrac{1}{2}$$

Secondly, you get no hit when all pointers land elsewhere, and that happens with probability $\left(\frac{11}{12}\right)^6$:

$$\text{Avge(B)} = -\text{probability of no hit} = -\left(\frac{11}{12}\right)^6 .$$

Adding these:

$$\text{avge(P)} = \frac{1}{2} - \left(\frac{11}{12}\right)^6 \approx 0.500 - 0.593 = -0.093 .$$

and the game indeed favours the house.

We can also see where Danny went wrong in his analysis. His assertion that on average, you get 6/12 = 1/2 dollar as payoff each play of the game was completely correct, and was exactly our payoff A. But his claim that you also get your own dollar back with probability 50% was in error. In fact, as we see above, you lose your dollar with probability 59.3% and so you get it back with probability only 40.7%.

The N-pad game. So what about the 13-pad game—the one that started it all? Of course it will favour the house even more than the 12-pad game. And what happens if we have less than 12 pads? When will we get a fair game, or a game that favours the player?—with 11 pads? 10 pads? Below I have tabulated the positive and negative components of the payoff for the 6-pointer game with various numbers of pads. We see that the 11-pad game is the closest to being fair, but still slightly favours the house. Only with no more than 10 pads does the positive part exceed the negative part, giving the player a positive net payoff.

6 pointers—different #'s of pads

# pads	*Avge(A) = avge # hits*	*Avge(B) = – prob. no hit*	*Payoff = avge(A)–avge(B)*
13	$\frac{6}{13} \approx 0.462$	$-\left(\frac{12}{13}\right)^6 \approx 0.619$	–0.157
12	$\frac{6}{12} \approx 0.500$	$-\left(\frac{11}{12}\right)^6 \approx 0.593$	–0.093
11	$\frac{6}{11} \approx 0.545$	$-\left(\frac{10}{11}\right)^6 \approx 0.564$	–0.019
10	$\frac{6}{10} \approx 0.600$	$-\left(\frac{9}{10}\right)^6 \approx 0.531$	+0.069

When there are 6 pointers, with 11 pads the net payoff is negative, giving an advantage to the house. However with 10 pads or less the player would have the advantage.

Problems

1.(a) You roll three standard dice. If you get at least one six, I give you a dollar; otherwise you give me a dollar. Does the game favour you or me?
(b) Same game except you roll four dice. Same question.

2. You pay me one dollar and then you roll six standard dice. I give you one dollar for each six that you roll. Does the game favour you or me?

3. You roll 6 standard dice and I give you $1 for every six that you roll. BUT, if you don't roll any sixes, you have to pay me $3. Who does this game favour?

4. You roll six standard dice and calculate the sum S. If S exceeds 21, I give you a dollar, if S is less than 21, you give me a dollar, and if S equals 21, it's a draw and no money is paid. Does the game favour you or me? [Can you use symmetry?]

5. You bet on any number between 1 and 6. Then three dice are thrown. You get $1 for every appearance of your number, but if your number fails to show you lose $1. Does the game favour you or the house? What is your average payoff?

6. I draw a card at random from a standard deck and then you get to roll that many standard dice (ace counts 1, king counts 13). If there's at least one six, I pay you 1 dollar; otherwise you pay me 2 dollars. Does the game favour me or you, and by how much?

7. I roll a standard die and whatever the outcome, you get to draw that many cards at random from a standard deck. If there's at least one spade, I pay you 2 dollars; otherwise you pay me 3 dollars. Does the game favour me or you, and by how much?

8. (a) Write down the general condition for the N-pad m-pointer game to favour the player.
(b) Prove that for all m the $2m$-pad m-pointer game favours the house. [[Hint: use the binomial expansion and then group the terms in pairs in a clever way.]

9. Look at the 6-pad 3-pointer game tabulated above. Notice an interesting pattern in the middle column. Use this to provide a simple proof that the column sum must be 1.

10. To make *Eiswein* you have to harvest the grapes as late as possible. As each day passes, the wine becomes more valuable, but there is always the possibility of a killer frost. Suppose that on Oct. 1 the value of the grape juice is $2/L, and it increases by $0.15/L per day throughout the rest of the month. Also suppose that there is a 3% probability of a killer frost on each October night, and once we have such a frost, the juice is worth only $1.50/L for cheap wine. When should the grapes be harvested?