## Pascal's Triangle

Pascal's triangle—here it is—at least the first ten rows:

$$
\begin{array}{ccccccccccccccccccccc}
&&&&&&&&&&1\\
&&&&&&&&&1&&1\\
&&&&&&&&1&&2&&1\\
&&&&&&&1&&3&&3&&1\\
&&&&&&1&&4&&6&&4&&1\\
&&&&&1&&5&&10&&10&&5&&1\\
&&&&1&&6&&15&&20&&15&&6&&1\\
&&&1&&7&&21&&35&&35&&21&&7&&1\\
&&1&&8&&28&&56&&70&&56&&28&&8&&1\\
&1&&9&&36&&84&&126&&126&&84&&36&&9&&1\\
1&&10&&45&&120&&210&&252&&210&&120&&45&&10&&1
\end{array}
$$

*An object of fascination to many high school students, most of whom know very little about it except the beguiling rule for generating it, and the fact that its entries often arise in unexpected places.*

"What is it?" I ask my students. "What is this triangle? What's it all about?"

They look back at me puzzled, but after a moment a few hands go up, and I collect three kinds of answers.

1) It's an array of numbers generated by a nice "sum" rule—put 1's on the outside and then each entry is the sum of the two numbers immediately above it (on each side).

2) The $n$th row is the set of coefficients in the expansion of the binomial $(1+x)^n$ .

3) The $n$th row is the set of combinatorial coefficients $\binom{n}{r}$ $(0 \le r \le n)$, the number of ways to choose $r$ objects from $n$.

These are all correct descriptions and it's worth taking some time to map their equivalence. We look at that now.

**Pascal's triangle.**
For convenience we take 1) as the <u>definition</u> of Pascal's triangle. These conditions completely specify it. The first row is a pair of 1's (the zeroth row is a single 1) and then the rows are written down one at a time, each interior entry determined as the sum of the two entries immediately above it.

**2) ⇔ 3)**
Why should the coefficients in the expansion of $(1+x)^n$ be the combinatorial coefficients? For example, in the expansion:

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

the coefficients 1, 5, 10, 10, 5, 1 are the quantities $\binom{5}{0}$, $\binom{5}{1}$, $\binom{5}{2}$, $\binom{5}{3}$, $\binom{5}{4}$, $\binom{5}{5}$. Why should this be? Why should the coefficient of $x^2$ be $\binom{5}{2}$?

**The combinatorial coefficient** $\binom{n}{r}$ (read as "$n$ choose $r$") is defined to be the number of ways to choose $r$ objects from a set of $n$. It is also written as $nCr$ or $C_{n,r}$.

Some scientific calculators have a combinatorial button (look on yours for one of the above notations). If you have one, calculate $\binom{5}{2}$ and check that you get 10, as promised in the above expansion.

Well let's start by asking what $(1+x)^5$ really means—it's the product of 5 copies of $(1+x)$:

$$(1+x)^5 = (1+x)(1+x)(1+x)(1+x)(1+x).$$

Now if we multiply this all out, we'll get lots of copies of each power of $x$, and the question is how many copies of $x^2$ will there be?

Well, a typical term in the expansion is got by taking one term (either 1 or $x$) from each of the 5 brackets. To get $x^2$, we have to take an $x$ out of two of the brackets, and a 1 from the other three. So there will be as many $x$'s as there are ways of selecting 2 brackets from 5 (these being the 2 brackets you take the $x$ out of). *That is, the coefficient of $x^2$ should be $\binom{5}{2}$ the number of ways of choosing 2 objects from 5.*

**3) ⇔ 1)** Why do the combinatorial coefficients appear as the entries of Pascal's triangle? The most elegant argument I know is *inductive*—it starts at the top of the triangle and verifies that we get the combinatorial coefficients in the first row, and then shows (quite dramatically, I feel) that the same rule that generates the triangle (the sum rule) also generates the combinatorial coefficients!

First let's look at the top of the triangle. The zeroth row (the single 1) is somewhat trivial (We say that $\binom{0}{0} = 1$ by definition, but I guess that makes sense) so we move to the first row, a pair of 1's. Again this is a bit trivial. We *define* $\binom{1}{0}$ to be 1 and certainly $\binom{1}{1}$ ought to be 1. We could move to the general inductive argument at this point but let's look at one more row. Row 2 has entries 1 2 1 and these are easily checked to be the numbers of ways of choosing 0, 1 and 2 objects from 2. Again $\binom{2}{0}$ is *defined* to be 1, and quite generally $\binom{n}{0}$ is defined to be 1.

Now we "move inductively down the triangle." We argue that if we have the set combinatorial coefficients for $n$ objects sitting along one row and we apply the sum rule, we'll get the combinatorial coefficients for $n$+1 objects. It's best to think with an example. Take the first 35 in row 7 which is "7 choose 3." Why should this be the sum of the 15 and the 20 in row 6 ("6 choose 2" and "6 choose 3"). That is, why should we have:

$$\binom{7}{3} = \binom{6}{2} + \binom{6}{3}?$$

A good exercise for the student is actually to count the number of ways of choosing 2 brackets from the 5. To be successful, such a count needs to be a bit organized. Here's one way to do it. Count all the ones that use the $x$ in the first bracket. There are 4 of those. Then the ones that are left that use the $x$ in the second bracket. There are 3 of those. And so forth. That approach actually gives us an interesting formula:

$$\binom{5}{2} = 4 + 3 + 2 + 1$$

For a generalization of this, see problem 7 below.

**Why is $\binom{n}{0}$ equal to 1?**

It makes intuitive sense—surely there's only one way to choose no objects from a set of $n$ and that's to choose no objects! But the case $r$=0 doesn't really fit the definition of $\binom{n}{r}$ and the fact that $\binom{n}{0}$=1 is really a convention, an agreement as to what it shall be. And why do we make that agreement? Because it gives us the algebraic properties that we want, the most important of which is the sum law which we are now establishing. For example:

$$\binom{7}{1} = \binom{6}{0} + \binom{6}{1}$$

This will clearly only be true if we set

$$\binom{6}{0} = 1.$$

That's the same reason by the way that we decide that $x^0$ should be 1—it works algebraically. In this case it's what we need to get the product law: $x^{n+0} = x^n x^0$.

Well there's a beautifully natural argument. Suppose you have a set of 7 different objects, and you want to choose a subset of size 3. Colour 6 of these objects black, and the remaining 1 white. The effect of this is to create two kinds of subsets of size 3—those that contain the white object, and those that don't. To make a set that contains the white object we need to choose 2 blacks from 6 and we can do that in $\binom{6}{2}$ ways, and to make a set that doesn't contain the white object we need to choose 3 blacks from 6 and we can do that in $\binom{6}{3}$ ways. So the total number of ways is the sum $\binom{6}{2}+\binom{6}{3}$. And that does it!

This is a lovely argument. I'm very fond of proofs like this in which a mathematical relationship is established using a clever "physical" interpretation of the terms of the formula. This is in fact an an important type of mathematical argument, but one with which most students have little experience. We used this approach in **Trains** to establish Fibonacci identities.

This works in general and gives us **Pascal's Formula**:

$$\boxed{\binom{n}{r}=\binom{n-1}{r-1}+\binom{n-1}{r}}$$

**Calculating $\binom{n}{r}$.**

Where are we? Having shown that the combinatorial coefficients can be found in Pascal's triangle we have a way to calculate them. For example, suppose we wanted to calculate the number of different possible bridge hands. That's the number of sets of 13 cards chosen from a deck of size 52, so that's $\binom{52}{13}$. To get that we just generate the first 52 rows of Pascal's triangle, and then start at the left end of the 52$^{nd}$ row and count to 13 (starting at 0).

But that just might be a lot of work (truly!). Is there a better way? The answer is yes—yes there is. There is a wonderful argument which produces a simple formula.

For example, suppose you want to choose 2 objects from $n$. Well you have to choose a first and then a second. Now there are $n$ ways of choosing the first and then, for each of these, $n$–1 ways of choosing the second (from the ones that are left!) for a total of $n(n–1)$ ways. But this method actually counts the *ordered* pairs—a "first" and a "second"—so each 2-set is actually counted twice, corresponding to the two orderings. So the number of different 2-sets is got by dividing by 2. And that gives us the formula:

$$\binom{n}{2}=\frac{n(n-1)}{2}.$$

This argument can be nicely generalized to give us a formula for $\binom{n}{r}$.

It's best to work with a specific example.

***Bridge hands***. Calculate $\binom{52}{13}$ which is the total number of different bridge hands.

The idea is to first count the *ordered* bridge hands, that is, suppose we were going to lay the 13 cards in a row, and distinguish two "hands" if they had the same cards but in a different order. Then how many ordered hands would there be? Well, in how many ways can I lay down 13 cards in order? Let's do it. There are 52 possibilities for the first card, and once it is down, there are 51 possibilities for the second, and then 50 possibilities for the third, etc. all the way to 40 possibilities for the 13th. So the number of such ordered hands must be

$$52\times51\times50\times...\times41\times40.$$

Now we consider that the above procedure counts any particular “unordered” hand many times, and we have to divide the expression by this number of times. So what is this divisor?—it is the number of ways of putting any particular hand into order. And how many ways are there of doing that? Well, this is the same argument again, but with a “deck” of size 13—there are 13 possibilities for the first card, and once it is down, there are 12 possibilities for the second, and all the way to 1 possibility for the 13th. So the number of such orderings is $13\times12\times...\times2\times1$. When we count ordered hands, every unordered hand is counted this many times, so the number of different unordered hands is the quotient:

$$\binom{52}{13} = \frac{52\times51\times\ldots\times40}{13\times12\times\ldots\times1} \approx 6.35\times10^{11}.$$

That's some 635 billion—enough to make it pretty unlikely you'll get the same one twice.

By the way, aren't you glad we didn't try to find this by generating 52 rows of Pascal's triangle?

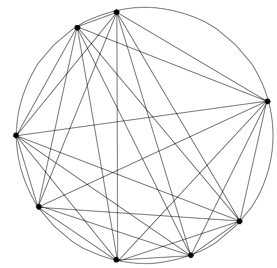
The general formula is established in the same way:

$$\boxed{\binom{n}{r} = \frac{n\times(n-1)\times\ldots\times(n-r+1)}{r\times(r-1)\times\ldots\times1}}$$

The reason I like working with this example is that I can bring a deck of cards to class, and actually display a typical hand, and "see" some of the re-orderings. And everybody likes cards

**The number of ways of choosing *r* objects from *n*.**
This formula has an important symmetry which helps us to remember how to write it down: the top starts at $n$, and the bottom starts at $r$ and they both “run down” for $r$ terms. Thus there are as many terms on the top as on the bottom. For example, for the formula for the number of bridge hands, the top starts at 52, and the bottom starts at 13, and they both run down for 13 terms

**Problems**



1.(a) Consider the picture at the right. How many different triangles are there all of whose vertices lie on the circle?
(b) How many different quadrilaterals are there with vertices all on the circle?

2. How many different bridge hands are there that contain only face cards (aces, kings, queens and jacks)?

3. Calculate the row sums of Pascal's triangle. What is the pattern? Find a combinatorial justification of this pattern.

4. Pascal's triangle is bilaterally symmetric. Find a combinatorial justification of this.

5. Using a $k\times(k+1)$ grid, find a geometric proof that

$$\frac{k(k+1)}{2} = 1+2+\cdots+(k-1)+k\ .$$

6. Recall our lovely argument for Pascal's formula: $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

We took $n$ objects and coloured 1 white and $n$-1 black and that allowed us to sorted the $r$-sets into two types. Here's an obvious generalization: for any $k<r$, colour $k$ of the objects white and $n$–$k$ black. Again, that gives us a way to classify the $r$-sets, and so will give us another formula for $\binom{n}{r}$. Might be interesting—check it out.

*Exploring the diagonals.* The set of 1's in Pascal's triangle that form the left hand edge is called the *zeroth diagonal*. Under that, is the *first* diagonal which goes 1,2,3, etc., and under that is the *second* diagonal, 1,3,6,10, etc. So that the $k$th diagonal consists of the numbers $\binom{n}{k}$. Now there are many nice patterns in the diagonals—how many can you find without reading further? Below we will study some of these.

7. *The partial sums of a diagonal.* The partial sums of any diagonal appear in the next diagonal. For example, the sum of the first four terms in the second diagonal is the fourth entry in the third diagonal:

$$1+3+6+10=20\ .$$

We can write this:

$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}=\binom{6}{3}\ .$$

Find a combinatorial explanation of this particular formula—this should allow us to see why it would hold in general. [Hint: Start by asking how you might calculate 6 choose 3. How might you partition the set of possible 3-sets?]

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

8. *Sums in the second diagonal.* Take the second diagonal and look at the pairwise sums:

1 + 3 = 4
3 + 6 = 9
6 + 10 = 16
10 + 15 = 25
etc.

1
1 1
1 2 **1**
1 3 **3** 1
1 4 **6** 4 1
1 5 **10** 10 5 1
1 6 **15** 20 15 6 1
1 7 **21** 35 35 21 7 1
1 8 **28** 56 70 56 28 8 1

Now why should the squares appear here? In fact there's a nice *geometric* argument. Take for example, the third sum, which can be written:

$$\binom{4}{2}+\binom{5}{2}=16\,.$$

Draw a 4×4 grid. Find "the right way" of partitioning the 16 cells into two subsets which produces the above formula. Explain how your construction generalizes to produce any formula in the above series.

9. *Differences in the third diagonal.* Take the third diagonal and look at the "second" differences:

10 – 1 = 9
20 – 4 = 16
35 – 10 = 25

1
1 1
1 2 1
1 3 3 **1**
1 4 6 **4** 1
1 5 10 **10** 5 1
1 6 15 **20** 15 6 1
1 7 21 **35** 35 21 7 1
1 8 28 **56** 70 56 28 8 1

Again, why should the squares appear here? Formulate this pattern using the combinatorial coefficients, and find an explanation for it similar in style to our argument for the Pascal's addition rule.

10. *Modified Pascal* If we replace the 1's along the right-hand side of Pascal's triangle by the powers of 2, and use the standard sum rule to generate the rest of the triangle, we get a new triangle. Show that the entries of the marked column are the powers of 4.

**1**
**1** **2**
1 3 **4**
1 **4** 7 **8**
1 5 11 15 **16**
1 6 **16** 26 31 **32**
1 7 22 42 57 63 **64**
1 8 29 **64** 99 120 127 **128**

[Hint: This is a wonderful little problem, and I don't want to give too much away, but I *do* want you to get started. It turns out that you can write the entries of this modified triangle in terms of the entries of the standard triangle in a way that gives you a nice combinatorial interpretation of the new entries. Look at the differences along each row. This problem is solved in *Math Horizons* Nov. 1994.]

11. Prove Pascal's formula algebraically using the formula $\binom{n}{r}=\dfrac{n\times(n-1)\times\ldots\times(n-r+1)}{r\times(r-1)\times\ldots\times 1}$.