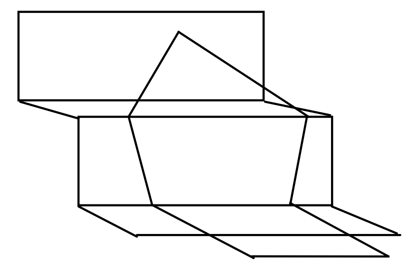
Polygons in Space

For which N is there a polygon in 3-space with N equal sides and all angles 90°? [Math. Mag. 55, p. 47, 1982]

For this session I bring a large supply of cut plastic straws and pipe cleaners for the 90° bends.

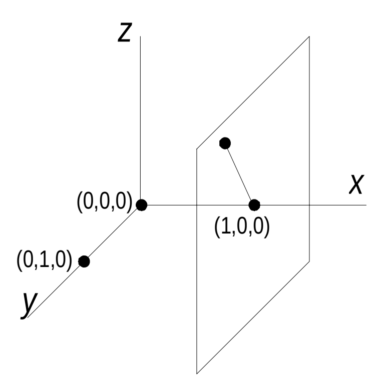
Students quickly discover the "staircase" model which provides examples for all even $N \geq 4$. By replacing the top of the staircase with a $(1,1,\sqrt{2})$ triangle, a model for $N = 7$ is discovered which generalizes, by lengthening the staircase, to all odd $N \geq 7$. The addition of the $(1,1,\sqrt{2})$ triangle requires spreading the top uprights, and care is required in the argument that the 90° angle is preserved at the base. And then the $(1,1,\sqrt{2})$ triangle has to be bent just the right amount. It's not easy to make 3-D arguments and it's a good exercise for the students to make things clear. It's useful to have the staircase as a template.



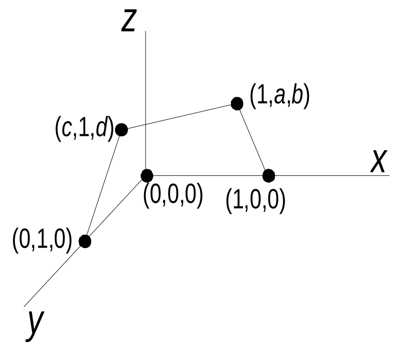
The problem is then $N = 5$. The students will spend a considerable time trying to build a model, and some will be convinced that they've virtually got it, just a little adjustment here and there... But it's hard to be sure you can get everything right. What sort of tools should we be working with here?

How about a geometric approach in the spirit of Euclid? Well, I've seen a couple of pretty good geometric arguments, but they're not easy—such arguments are often difficult in 3-space. A much more hopeful approach is to use analytical geometry, that wonderful invention of Rene Descartes (1596-1650). The idea here is to introduce a system of coordinates and keep track of the vertices of the polygon with their coordinate triples. If we can write down a system of equations which expresses the condition that the angles be 90°, then we have reduced our geometric problem to the unglamorous(?) but reliable(?) world of algebra.

How to begin? Let's suppose we have constructed such a 5-gon. It's always a good idea to position and orient our figure so that the coordinates stay as simple as possible, so let's put one vertex at the origin, and the two adjacent sides along the x- and y-axes. Thus, the coordinates of our first three vertices are (0,0,0), (1,0,0) and (0,1,0). Now what can we say about the remaining two vertices? Well, the edge adjacent to (1,0,0) must be perpendicular to the x-axis, so must lie in the plane perpendicular to that axis passing through (1,0,0). What does that tell us about the coordinates of the far end of that edge? That's a simple enough question, but the students will need a moment to think about it.



In fact the plane we are after is the plane of all points whose *x*-coordinate is 1. In fact its equation is *x*=1. So the other end of this edge must have *x*-coordinate 1, that is, it must have coordinates (1,*a*,*b*) for some numbers *a* and *b*. A similar argument on the other side, with the *y*-axis, shows that the other vertex must have the form (*c*,1,*d*) for some numbers *c* and *d*.



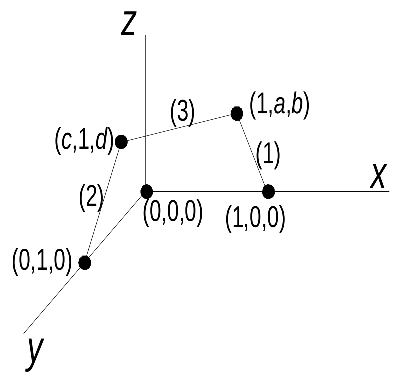
So the question becomes, can we choose the 4 "unknowns" *a*,*b*,*c*, and *d*, to get a polygon of the required form. Well, what are the requirements still to be satisfied?

First, all sides must have length one. So the distance between every pair of adjacent vertices must be 1. The two sides along the *x*- and *y*-axis were chosen to have length one, so there remain three conditions for the other three sides. We have to recall the formula for the distance *D* between two points (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$D = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

Many students will not have seen this formula, but they should be familiar with the 2-D analogue, and can certainly construct it from Pythagoras.

The three equations we get are:



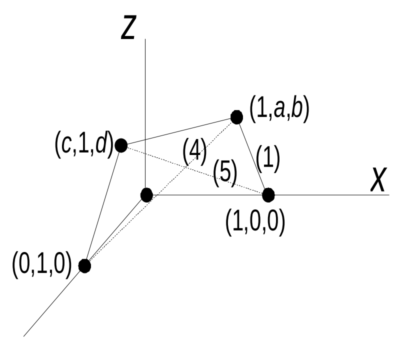
(1,*a*,*b*)-(1,0,0,): $a^2 + b^2 = 1$ (1)

(*c*,1,*d*)-(0,1,0,): $c^2 + d^2 = 1$ (2)

(*c*,1,*d*)-(1,*a*,*b*): $(c-1)^2 + (1-a)^2 + (d-b)^2 = 1$ (3)

Now we have to get the angles right (pun!). The angle at the origin is okay because we put the adjacent sides along orthogonal axes. And we also set the two angles adjacent to these to be 90°. That leaves us with two angles to set.

But what is the algebraic condition for a right angle? Well, one can get involved with vectors and dot product, but given the above distance formula, it's much easier to use distances: a triangle with two sides of length 1 has a right angle between them if the third side has length $\sqrt{2}$ (Pythagoras). This means that the distance between (1,0,0) and (*c*,1,*d*) must be $\sqrt{2}$, and the distance between (0,1,0) and (1,*a*,*b*) must also be $\sqrt{2}$. The equations are



(1,*a*,*b*)-(0,1,0,): $1^2 + (a-1)^2 + b^2 = 2$ (4)

(*c*,1,*d*)-(1,0,0,): $(c-1)^2 + 1^2 + d^2 = 2$ (5)

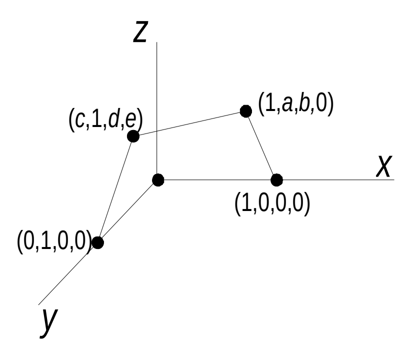
These five equations are quite simple, and it is not hard to show that they are inconsistent. In fact, the students will vie quite readily to find the most elegant path to this conclusion. Thus we have a proof that *N*=5 is impossible.

Four dimensions. Now for an interesting problem. Would N=5 work if we lived in a four-dimensional world?

The initial reaction of the students to this question is that they could never solve such a thing. In fact many of them will protest that they don't really understand what you mean.

It's helpful to think of two-dimensional creatures, who live in a plane. Suppose they were given the polygon problem. They certainly know what an edge of length one is, and they understand right angles, so they would understand the problem. They wouldn't find it very interesting however, because they'd see right away that there is no solution except for N=4. So then if you asked them what the solution would be in 3-space, they'd have the same problem visualizing things that you do for 4-space, and a geometric solution would be out. ***But they could understand the above algebraic solution!*** So they'd be able to prove that it was impossible for N=5. In the same way, using the coordinate approach, we should be able to settle the 4-dimensional question.

Let's see how it goes. We now have general coordinates (x,y,z,w). Just as before we choose the first three vertices to be (0,0,0,0), (1,0,0,0) and (0,1,0,0). Now, for the edge that goes "up" from the x-axis to be perpendicular to that axis, it must have x-coordinate 1, and the other three coordinates could be anything (instead of having a plane of points perpendicular to the x-axis, as before, we now have a 3-dimensional "hyperplane" of such points). Now I assert that we can always arrange for this fourth vertex to have last coordinate 0. Indeed, the three edges we have so far, will lie in a three-dimensional subspace and we can suppose that it is the subspace w=0. So this fourth vertex will have the form $(1,a,b,0)$. The final vertex will have the form $(c,1,d,e)$.



The equations are the same as above except an e is added to the left-hand side of (2), (3) and (5).

$(1,a,b)$-$(1,0,0,)$: $\quad a^2+b^2=1 \qquad (1)$

$(c,1,d)$-$(0,1,0,)$: $\quad c^2+d^2+e^2=1 \qquad (2)$

$(c,1,d)$-$(1,a,b)$: $\quad (c-1)^2+(1-a)^2+(d-b)^2+e^2=1 \qquad (3)$

$(1,a,b)$-$(0,1,0,)$: $\quad 1^2+(a-1)^2+b^2=2 \qquad (4)$

$(c,1,d)$-$(1,0,0,)$: $\quad (c-1)^2+1^2+d^2+e^2=2 \qquad (5)$

Now we have 5 equations in 5 unknowns, and there is a solution, which is essentially unique (up to sign) with

$a=c=1/2$, $\quad b^2=3/4$, $\quad d=1/2b$, $\quad e^2=5/12$

So it's possible!

Problems
These problems are all amenable to a coordinate geometry attack.

1. This is essentially the same problem of this section with 90° replaced by 60°. One can imagine all sorts of complicated problems of this type, but this one has some nice simplicity about it.

(a) Show that there are quadrilaterals and hexagons in 3-space with equal sides and all angles equal to 60°. This can be done geometrically—just look at the right model!

(b) But are there pentagons with this property? Examine this question algebraically by introducing coordinates. The numbers will come out nicer if you take the common side-length to be 2. In this case you will wind up with 7 equations in 6 unknowns. They are not at all hard to unravel, but handle them carefully. In particular, when you take square roots, don't forget the $\pm$.

2. (*From Edgar Allen Poe*) A young man found among his great-grandfather's papers the location of a hidden treasure:

"Sail to 16.7 latitude and 175.2 western longitude where you will find a deserted island. On the island is a large meadow with a lonely oak and a lonely pine tree. There is also an old gallows. Start from the gallows and walk to the oak counting your steps. At the oak turn left by a right angle and take the same number of steps. Put a spike in the ground. Now, return to the gallows and walk from there to the pine counting your steps. At the pine turn right by a right angle and take the same number of steps. Put another spike in the ground. Dig half-way between the spikes, the treasure is there."

The young man found the island, the meadow, the oak and the pine, but to his sorrow, the gallows was gone. He started digging at random all over the island. But all his efforts were in vain, the island was too big. So he sailed home empty-handed. Had the young man known some mathematics, he could have found the treasure. How?

3. (*A wonderful exploratory problem that we use with grade 11 students.*) Two tall trees stand at A and B a considerable distance apart in the middle of a flat plain. The tree at B is twice the height of the tree at A. There is a road on the plain and as I cycle along it I am struck by the fact that the two trees appear to be exactly the same size. As the road moves towards them, the trees both appear to get larger, but they continue always to look the same size as one another. *Are you ready for the question?*—what's the shape of the road?

4. *Cheating.* In the early part of the investigation, many of the students made a model of the 5-gon that they were convinced was right. In fact of course it wasn't quite right but it was certainly close. So that raises the question of how close you can yet. That is, suppose you are allowed to "cheat" and have the some of the side lengths not quite 1 or some of the angles not quite 90°, what can you do? Or more precisely, how close can you get? In this form, the question is slightly unstructured, but here are two simple versions.
(a) Suppose the constraint is that four of the sides must have length 1 and all angles 90°. What are the possibilities for the fifth side? [Take care in setting this up. Don't make the algebra more difficult than necessary.]
(b) Suppose the constraint is that four of the angles must be 90° and all sides must have length 1. What are the possibilities for the fifth angle? [Again take care in setting this up. Work in the first instance with the length of one of the diagonals, which is allowed to be different from 2.]