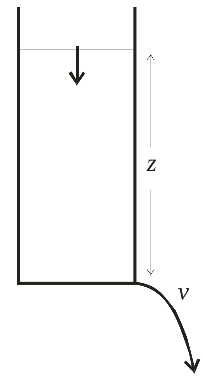
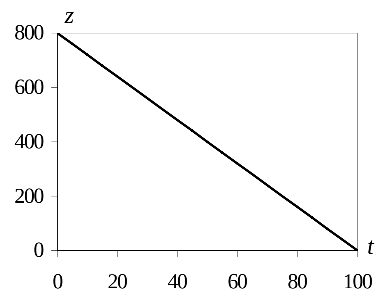
10. Water tank



The graph
A cylindrical tank contains 800 ml of water. At t=0 (minutes) a hole is punched in the bottom, and water begins to flow out. It takes exactly 100 seconds for the tank to empty.

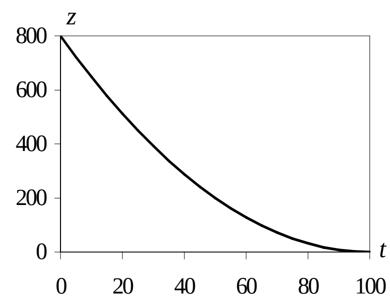
Example I. Draw the graph of the amount z of water in the tank against time t.. Explain the shape of the graph.

I begin, of course, by letting the class watch the action. I use a 1 litre water bottle or pop bottle, even though this is not quite cylindrical, with a hole that's 3 to 4 mm in diameter. We don't take measurements or record data—that comes later.



I invite a student to come and draw the graph on the board and interestingly enough, what I get is a straight line. I ask the class what they think and there is general cautious nodding. What's good about the graph? It starts at height 800, decreases, and hits the t-axis at 100. That seems to fit the given information. So we take a vote. How many agree? I get a forest of hands. Holy cow. That really surprises me.

Fortunately there *are* a couple of objectors. *Why is it straight? Shouldn't it be a curve?*



This student comes up and draws a new picture, and I get something like the graph at the right. Okay. What's better about this graph? Why should it be curved?
Well the flow rate is higher at the beginning than at the end.
Why is that?
More water in the tank—more pressure pushing the water out.
So how does that relate to the geometry of the graph?
Well, when the flow rate is high, z goes down fast, and the graph is steep. And then when the flow rate is less, the graph is flatter.

How does the decrease in the flow rate over time relate to the curvature of the graph? It's important that they are able to explain that in an intuitive manner.

That's good, and this time it's not hard to get the class to agree that it's right.

Okay—what might we have done to get that straight-line graph?
We'd have to take the water out at a constant rate.
How might we do that?
Maybe pump it out at a constant rate?
That would do it. What constant rate would the pump use?
800 ml in 100 seconds means 8 ml/s.
Right.

The formula.

Example 2. Finding an equation for the graph.

I ask the class if they can produce an equation for a graph with the right shape, and of course they give me a parabola. This turns out to be correct on theoretical grounds.

Theoretical considerations of how the pressure pushes the water out of the hole lead in fact to the equation of a parabola. A simple-minded argument for this is not so easy to find, but I give some heuristic considerations at the end. It's neat that a potentially complex phenomenon like this leads to a simple equation. In fact the physical world is full of similar surprising stories.

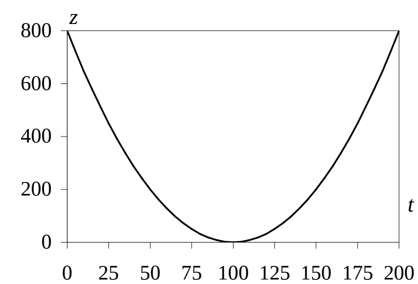
Given that it's a parabola, I challenge them to find its equation. They should use the given data—that the tank starts with 800 ml of water and that it takes 100 seconds to empty.

Well the equation of a parabola is a polynomial of degree 2, and many of them start with the algebraic form:

$$z(t) = at^2 + bt + c$$

leaving them three parameters, a, b and c to evaluate. And they can't quite do that because they've only two conditions to work with, that $z(0)=800$ and $z(100)=0$. So they're stuck..

I suggest that they draw the whole parabola (of which only the left half belongs to the water tank problem), and then see what they can say about the equation. When they do that, they realize that the vertex of the parabola must be at t=100.



Why is that? *Because the flow rate out of the tank determines the slope of the graph and as the tank becomes empty, the flow rate must fall to zero, and therefore the slope at the end point must be zero*.

Okay. How do we use that algebraically? Well, when we know the vertex of a parabola, there's another algebraic form that's better to use—the completed square form—because the coordinates of the vertex both appear among the three parameters:

The completed square form

$$\mathbf{y = a(x-c)^2 + b}$$

If you are working with a parabola, this is actually a wonderful algebraic form to be given. The parameters a, b and c all describe features of the graph which are geometrically and physically significant. "c" and "b" give you the location of the vertex ($x=c$ and $y=b$), and "a" tells you how fast the parabola "goes up."

In this case the vertex of the parabola is at (100,0) and hence the equation has the form:

$$z = a(t-100)^2 .$$

To find a plug in the initial condition $t=0$, $z=800$:

$$800 = a(0-100)^2 = 100^2 a$$

and this solves to give $a=800/100^2 = 0.08$. Thus the equation of the parabola is

$$z = 0.08(t-100)^2 .$$

Experiment.

It's time to fill up the tank, open the hole, take some careful readings, and see if we can verify the parabola result. What we have to keep track of is the volume z in the tank, but for a cylindrical tank, volume is proportional to height, so we used "height above the hole" as our measure of amount. In fact I have drawn horizontal marks at intervals of 10 mm above the hole, all the way from 10 to 130 and then I have the students record the time that the level arrives at each of the marks.

Now we have to decide whether these points lie on a parabola. They certainly have the right shape, but there are any number of curves around with that shape, so we need some mathematical way of checking it out. What we do know is the "target" shape that we're looking for:

$$z = a(t-c)^2 .$$

Our job is to see if there are any values of the parameters a and c for which the curve provides a good fit to our data, and if so, what are the "best-fit" values of these parameters.

I ask the class for suggestions. One idea is that we take the two data points we used in the previous section—the beginning (0,130) and the end (192,0)—and then use these to find the values of a and c, and then maybe we could plot the equation on top of the points and see how good the fit was.

That's an interesting idea, but it only uses the information contained in two of our data points. What if either of those points happened to be off? The point is that there will always be errors of measurement in experimental data, and we need curve fitting techniques that are robust in the sense that they're not going to be seriously compromised by one or two bad points. Indeed, some of our points will be better than others, but it's not clear at first which the "good" ones are, and we need methods which can rise above that.

But the most popular idea is to just *fit a parabola to the data*. Technology has come a long way in 10 years most spreadsheets and hand calculators will allow you to fit a "trendline" of a variety of algebraic forms, not only linear but polynomial of any degree or exponential etc. And the kids know how to use that.

But I want to be able see "with my own eyes" whether these points lie on a parabola, and what my eyes are good at seeing are straight lines. So I want to somehow "transform" the data so it ought to lie in a straight line.

What kind of transformation would do that? Well z is quadratic in t. What about looking at its square root?

The experiment I record here used a 1-litre pop bottle. I avoid the non-cylindrical part at the top and the bottom by not filling the bottle right to the top, and by punching the hole in the recessed centre of the bottom so that a residue of water sits permanently in the bottom of the bottle.

height z (mm)	time t (s)
130	0
120	7
110	15
100	24
90	33
80	43
70	53
60	64
50	76
40	90
30	105
20	125
10	149
0	192

Careful with the end points. Interestingly enough, the end point t=192, z=0, is apt to be even more prone to error than the others. When the flow is very weak, effects like surface tension, which our theoretical model does not account for, can become significant, and cause the flow to stop prematurely.

In fact, read on!

The elementary approach. This is of course what we would have done when I was at school. It's unsophisticated, but there's a lot of good learning in it. In transforming the data to a linear form, what we are doing is what all those sophisticated spreadsheets are doing behind the scenes.

Transforming the data.

The "target" equation

$$z = a(t-c)^2$$

is quadratic in t. But we can get rid of the square by taking square roots of both sides:

$$z^{1/2} = \pm a^{1/2}(t-c) .$$

It's important to remember the plus and minus signs, because it's not at first clear which of the two we want. Let's think about this. There are only certain times that are relevant and those are the times between 0 and c. If we look at the sign of both sides for these times, we see that everything's positive except the $(t-c)$, *which is negative*. So to make both sides positive in that interval, we need to choose the minus sign:

$$z^{1/2} = -a^{1/2}(t-c) .$$

Actually it would read better if we turned the $(t-c)$ term around to read $(c-t)$ and then that term would be positive for the relevant t-values:

$$z^{1/2} = a^{1/2}(c-t) .$$

Now all three terms are positive for the relevant values of t.

So where have we got to? Do we have a straight line yet or not? Well the expression is now linear in t, but not in z—we don't have the square anymore, but we have a square root instead.

However, the expression is linear in $z^{1/2}$. Is that any use? Yes it is. That tells us that *if we were to tabulate the values of the square root of z, and plot those against t, we should get a straight line*. Well, well.

And we do! There's a real sense of pleasure and excitement in watching that straight line of points emerge. That's the "proof" we were looking for that the original data points lie along a parabola.

Now we can ask for a linear trend line. We get the equation.

$$z^{1/2} = 11.415 - 0.0573t .$$

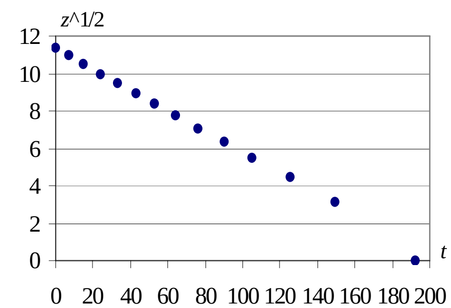
A student remarks that the last data point is below the line and farther from it than any other. It would appear that our experiment ended (i.e. the flow stopped) a few seconds before it "should" have. Does that make any sense?

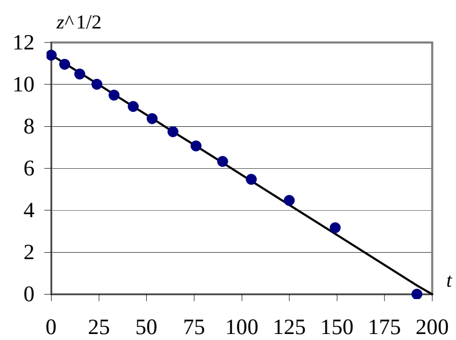
I throw this question out and get a good discussion. This is a graphic illustration of the surface tension effect, the low pressure at the end being insufficient to overcome the viscosity. So that in fact a small amount of water remained above the hole at the end.

This is the first tiny opening of a window onto a sophisticated and powerful idea.

We are changing variable. The new variable, $z^{1/2}$, is much "friendlier" for our purposes because if z has the target algebraic form, then the $z^{1/2}$-data will be linear.

t	z	$z^{1/2}$
0	130	11.4
7	120	11.0
15	110	10.5
24	100	10.0
33	90	9.5
43	80	8.9
53	70	8.4
64	60	7.7
76	50	7.1
90	40	6.3
105	30	5.5
125	20	4.5
149	10	3.2
192	0	0



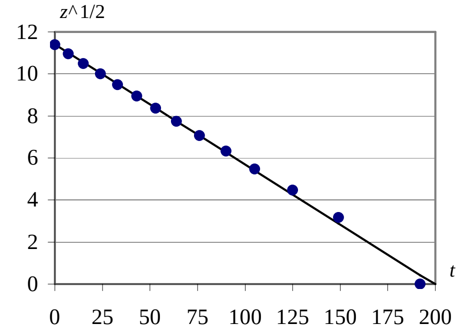


Analysis

So what do you think of the line? Is everything okay?

Most students seem happy enough. But one or two are uneasy. The points between t=100 and t=150 are all above the line. The problem is clearly caused by the last point. Being so "low" it has pulled the line down at the end. *It would be better if it weren't there!*

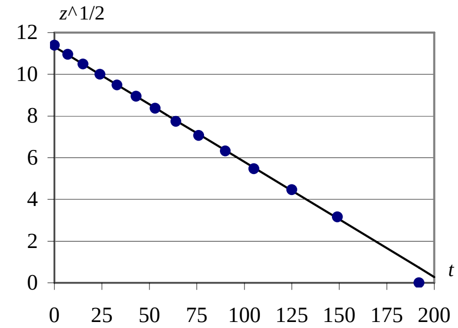
Quite so. Indeed we've already identified that last point as anomalous, and we even know the reason—viscosity effects. So we have a good reason to simply exclude it from the data set.



Let's do it. So we order up another trend-line using only the first 13 points:

$$z^{1/2} = 11.321 - 0.0552t$$

and indeed it has a slightly flatter slope. And it gives a much better fit. So this is the line we will use.



What we can do now is "untransform" this equation back into an equation for z. Square both sides:

$$z = (11.321 - 0.0552t)^2.$$

There it is. Actually, the form we prefer it to be in is the above form $z = a(t-c)^2$ because then the time c to empty is one of the parameters. How do we put it in that form? Well we want t to have coefficient 1, so we take the coefficient of t outside the bracket.

$$z = \left[0.0552\left(\frac{11.321}{0.0552} - t\right)\right]^2$$

$$= (0.0552)^2\left(\frac{11.321}{0.0552} - t\right)^2$$

$$z = (0.0552)^2\,(205-t)^2.$$

According to the equation as it is now written, the "theoretical" time that the tank should've been empty (if there had been no surface tension effects) is t=205. This is the point where the regression line meets the t-axis (just off the right end of the diagram). It appears that the flow stopped 13 seconds early. Evidently there was some water at the end that the viscosity trapped inside.

Also by setting t=0 we find that the starting water level is

$$z_0 = (0.0552)^2\,(205) = 128 \text{ mm}.$$

That's to the z_0 = 130 that we aimed for. Maybe we were a tad short at the beginning. Or maybe not…

This completes the "proof" that the z-points lie in a parabola. The $z^{1/2}$-data was a straight line, and by squaring the equation of that line, we get a parabola.

Most of my students have trouble with these manipulations—more trouble than I would like to think they would have. I feel that it's not so much that they don't know how to do the steps, as they have not learned that great care is needed with such calculations

Problems.

1. A parabola has vertex at x=4, y=2. It intersects the y-axis at y=34. Make a sketch of its graph and find its equation.

2. At t=0 (minutes) a tank holds 1000 L of water. Over the next 5 minutes a total of 500 L is pumped out of the tank at a variable rate so that at t=5 the tank holds only 500 L. The equation of the amount z (litres) in the tank at time t (minutes) has the form

$$z = a(5-t)^3 + b.$$

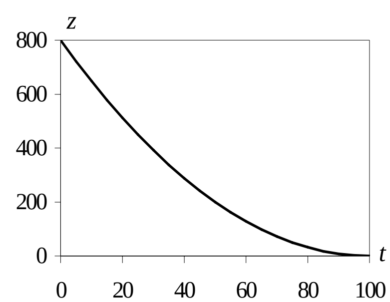
Find the parameters a and b.

3. A tank contains has 200 litres of water. Suppose we pump the water out of the tank at the constant rate of 20 L/min. Draw the graph of the amount z in the tank against time t over the 10-minute period until the tank is empty and find the equation of this line. Take the time origin t=0 to be the moment at which the pump begins to operate.

4. A cylindrical tank starts with 800 litres of water. Suppose we pump water out of the tank for 10 minutes at the constant rate of 20 L/min and then for another 10 minutes at the constant rate of 40 L/min.
(a) How much water will there be in the tank at the end of the 20-minute period?
(b) Draw the graph of the amount z in the tank against time t over the 20-minute period.
(c) Find the equation of the line segments.
(d) At what time is the tank half full (i.e. 400 litres)? Illustrate this on a copy of your graph.

5. A cylindrical tank has 900 litres of water. At t=0 (minutes) a hole punched at the base, and water pours out. It takes exactly 60 minutes for the tank to empty. Use the fact that the z-t graph is a parabola to answer the following questions.
(a) Find a formula for z as a function of t.
(b) How much water is in the tank at t=30 minutes?
(c) At what time are there 400 L in the tank?
(d) When is the tank half-full?

6. At the right is a copy of the parabola that results from Example 1 of this section. Suppose the water which runs out of the hole is directed into a bucket which starts empty. Draw the graph of the amount of water in the bucket as a function of time t. What is the geometric relationship between the curve that you have drawn and the graph at the right? Be as precise as you can. Illustrate by including both curves in the same diagram.



7. In Example 1 we preferred to work with an equation of the form $z = a(c-t)^{1/2}$ because the parameter c has a nice physical interpretation as the time for the tank to empty. But what about the parameter a? Well there isn't a nice interpretation for it, but there is another physically significant parameter, and that's the starting height of the water, call it h. Here's the question: find a form of the general equation which uses these two nice parameters c and h, instead of a and c. [And that's actually the "natural" form to work with.]

8. A tank with a hole in the base is empty. Starting at time t=0, water is run into the tank at a constant rate. After a reasonable time has passed, the water level in the tank seems to be constant at depth 100 mm. Draw the graph of the water level in the tank as a function of time for $t \geq 0$.

9. Two identical cylindrical tanks, A and B each contain 800 ml of water. One hole is punched at the base of A and two holes are punched at the base of B, all three holes of the same size. On a single set of axes, draw graphs A and B of the amounts of water in each tank against time. What relationship exists between the slopes of graphs A and B? Illustrate this relationship by referring to specific points on the graphs.

10. I have two cylindrical tanks, A and B, the first with twice the cross sectional area of the second. The tanks have identical holes at the base, and each starts with 800 ml of water. On a single set of axes, draw graphs A and B of the amounts of water in each tank against time. What relationship exists between the slopes of graphs A and B? Illustrate this relationship by referring to specific points on the graphs.

11. The data set at the right is supposed to fit a model of the form

$$z = a(c-t)^{1/2}$$

but there are random errors in the collection and no reliable figure was obtained for the t-value when z=0. Transform the data to fit a straight-line model, and use ruler and eye or a trend-line routine to plot the data, obtain the best-fit straight line, and estimate the parameters a and c. What's your best guess for the time at which z=0?

t	z
0	60
10	57
20	55
30	52
40	49
50	46
60	43
70	40
80	37
90	32
100	28
110	22
120	15
?	0

12. I have some x, y data which are supposed to fit an equation of the form $y = a(x-b)^2$ for some suitable values of the parameters a and b. So I plot $y^{1/2}$ against x and I get a fairly good straight line. The regression equation is $y^{1/2} = 8.32-0.20x$. What estimates of a and b does this line gives us?

13. I have some x, y data which are supposed to fit an equation of the form $y = a(x+b)^3$ for some suitable values of the parameters a and b. So I plot $y^{1/3}$ against x and I get a fairly good straight line. The regression equation is $y^{1/3} = 2.45+0.05x$. What estimates of a and b does this line gives us?

14. A large cylindrical tank containing 100 litres of water empties through a hole at the base. The data set at the right shows the time t (minutes) at which the water level is at different heights z (cm). Thus the data are supposed to fit a model of the form $z = a(c-t)^2$. Transform the data to fit a straight-line model, and use ruler and eye or a spread sheet and a regression routine to plot the data, obtain the best-fit straight line, and estimate the parameters a and c.

t	z
0	100
18	90
36	80
54	70
75	60
96	50
121	40
150	30
179	20
215	10
273	0

15. The graph of $z^{0.7}$ against t is a straight line with slope –1.3 and t-intercept t=42. Find an equation for z in terms of t.

16. I have three cylindrical tanks, one with 100 L, one with 200 L, and one with 300 L. Now I punch holes at the base of each of them and let them all start to empty at the same time, the first two empty into pool A and the third empties into pool B. Remarkably enough, all three tanks becomes empty at the same time. If both pool A and B start empty, do they always have the same amount of water?