6. The factor theorem

Example 1. At the right we have drawn the graph of the polynomial

\[ y = x^4 - 9x^3 + 28x^2 - 36x + 16. \]

Your problem is to write the polynomial in factored form. Does the geometry of the graph give you any help here?

Solution

If this graph had appeared in the previous section, we'd have expected it to belong to an equation which was a product of linear factors corresponding to the zeros \(x=1,2,4\). That is, we'd expect to have the factors \((x-1), (x-2)\) and \((x-4)\). Actually, since the graph does not cross the axis at \(x=2\), we'd expect \((x-2)\) to occur an even number of times, perhaps twice.

Well, if we have one copy of \((x-1)\), two copies of \((x-2)\) and one copy of \((x-4)\), that's already a product of degree 4 and the whole polynomial has degree 4, so that must be it. So we propose:

\[ y = (x-1) (x-2)^2 (x-4). \]

Is this the answer?

Let's check it out. What we do is expand the factored form and see if we get the original.

\[
\begin{align*}
y &= (x-1)(x-4) \star (x-2)^2 \\
&= (x^2 - 5x + 4) \star (x^2 - 4x + 4) \\
&= x^4 + (-4-5)x^3 + (4+4+20)x^2 + (-20-16)x + 16 \\
&= x^4 - 9x^3 + 28x^2 - 36x + 16.
\end{align*}
\]

which is the original equation.

So we did it! We used the graph to factor the polynomial. The graph gave us the zeros and the zeros gave us the factors.

Was it obvious that this was going to work? Well not quite. But there's a theorem that says that it always will. The theorem isn't obvious and it has to be proved, but the proof is a nice argument, and we give a version of it below. What the theorem says, roughly speaking, is that if you have a zero of a polynomial, then you have a factor.

In the second line of this calculation we have the product of two terms each with 3 entries. To expand this we multiply each entry in the first term by each entry in the second. That gives us \(3 \times 3 = 9\) terms and these are displayed on the third line. There's 1 \(x^4\)-term, 2 \(x^3\)-terms, 3 \(x^2\)-terms, 2 \(x\)-terms, and 1 constant term.

For example, the three ways to get an \(x^2\) term are diagrammed below:

\[
\begin{align*}
4 & \quad 4 \\
\hline
(x^2 - 5x + 4) & (x^2 - 4x + 4) \\
20 & \hline
\end{align*}
\]
The factor theorem.

**Geometric version.** If \( f(x) \) is a polynomial whose graph crosses the x-axis at \( x=a \), then \( (x-a) \) is a factor of \( f(x) \).

**Algebraic version.** If \( f(x) \) is a polynomial and \( f(a) = 0 \), then \( (x-a) \) is a factor of \( f(x) \).

Proof of the factor theorem

Let's start with an example. Consider

\[
f(x) = x^4 - 9x^3 + 28x^2 - 36x + 18.
\]

[This is the polynomial of Example 1 with last term 18 instead of 16.] Our question is whether or not it has \( x-4 \) as a factor.

Well, one way to find that out is to divide it by \( x-4 \) and see what we get. Either it will go evenly or there will be a remainder. If we do this (perhaps by long division of polynomials) we find:

\[
x^4 - 9x^3 + 28x^2 - 36x + 18 = (x-4)(x^3 - 5x^2 + 8x - 4) + 2
\]

which has the form:

\[
f(x) = (x-4)Q(x) + r
\]

for a quotient \( Q(x) \) and a remainder \( r \). The remainder here is 2, which is not zero, and thus we can conclude that \( x-4 \) does not divide evenly into \( f(x) \).

Now here's the interesting result—that remainder 2 is the value of \( f(x) \) at \( x=4 \). You can verify that by plugging \( x=2 \) into the expression for the polynomial \( f(x) \), but an easier and more insightful argument is obtained by plugging \( x=4 \) into the above equation. We get:

\[
f(4) = (4-4)Q(4) + r
\]

and the stuff in the middle is zero, giving us \( r = f(4) \). So we can write the above equation as:

\[
f(x) = (x-4)Q(x) + f(4).
\]

The reason this “fast” argument is so nice is that it works in general. We can do this for any polynomial \( f(x) \) and for any factor \( (x-a) \). When we perform the above routine, we get:

\[
f(x) = (x-a)Q(x) + f(a).
\]

And just as above, the remainder when \( f(x) \) is divided by \( (x-a) \) is \( f(a) \). It follows from this that \( (x-a) \) will be a factor of \( f(x) \) exactly when \( f(a)=0 \). And that’s the factor theorem.
Example 2. Consider the polynomial
\[ f(x) = x^3 + 4x^2 - 17x - 60. \]
Verify that \( f(-3) = 0 \), and use this observation to find the other two zeros of the polynomial.

Solution
First we show that \(-3\) is a zero:
\[ f(-3) = (-3)^3 + 4(-3)^2 - 17(-3) - 60 = -27 + 36 + 51 - 60 = 0. \]

Now how might we find the other zeros? Well, that’s where the factor theorem comes in. The zero \(-3\) gives us the factor:
\[ (x + 3). \]

If we divide this factor into \( f(x) \), we’ll get a quotient of degree 2. That’s a quadratic polynomial and we can find its zeros either by factoring it or using the quadratic formula. Let’s do it.

To divide \((x+3)\) into \( f(x) \), we can use long division, but in simple cases, trial and error is just as easy. We write:
\[ x^3 + 4x^2 - 17x - 60 = (x+3)(x^2 + \underline{\phantom{1}}x - 20) \]

We have filled in the first and last terms of the quotient. Both of these are easy to obtain. We get the \(x^2\) by seeing what has to go with the \(x\) to make \(x^3\), and we get the \(-20\) by seeing what has to go with the 3 to make \(-60\). So all that remains is to fill in the \(x\)-term, and we have written its coefficient as an underlined blank.

We can do this in two way, choosing it so as to get the \(4x^2\) right or choosing it so as to get the \(-17x\) right. Let’s go for the \(4x^2\). We already have \(3x^2\) from the 3 and the \(x^2\) so we need one more, and that means that the \(x\) should be 1. Then that will go with the \(x\) in the \((x+3)\) term to give \(1x^2\). Thus the factorization is
\[ x^3 + 4x^2 - 17x - 60 = (x+3)(x^2 + x - 20). \]

The remaining two zeros will be zeros of the quadratic factor. We can use the quadratic formula, or factor it. In fact it factors:
\[ x^2 + x - 20 = (x+5)(x-4). \]

Thus the entire factorization is:
\[ x^3 + 4x^2 - 17x - 60 = (x+3)(x+5)(x-4) \]
and the zeros are \(-3, -5, \) and 4.

This is a bit of a drawn out presentation, but here’s the scoop.

1. Since \(-3\) is a zero of \( f(x) \), \( x+3 \) is a factor.

2. Divide \(x+3\) into \( f(x) \) to get the factorization:
\[ (x+3)(x^2+x-20) \]

3. Factor the second term to get
\[ (x+3)(x+5)(x-4). \]

4. The zeros are 
\(-3, -5, \) and 4.

As a check on our work, we see if the \(x\)-term is right.
We need \(-17\) of these. The above factorization gives us
\(-20 + 3 = -17\, \text{as expected.} \)
Example 3. Find all solutions of the equation:

\[ x^3 - x^2 - 6x - 4 = 0. \]

Solution
This time we are not given a zero of the polynomial. One idea is to plug a few numbers in, hoping that by some stroke of luck, a small integer (like ±1 or ±2) might work. We try:

\[
\begin{align*}
 f(1) &= (1)^3 - (1)^2 - 6(1) - 4 = -10 \\
 f(-1) &= (-1)^3 - (-1)^2 - 6(-1) - 4 = 0
\end{align*}
\]

and −1 works. [I guess we're lucky after all.] It follows from the factor theorem that \((x+1)\) is a factor. We divide this into \(f(x)\) and get

\[ x^3 - x^2 - 6x - 4 = (x+1)(x^2 - 2x - 4). \]

The quadratic quotient \(x^2 - 2x - 4\) will not factor nicely, so we use the quadratic formula to find its zeros:

\[ x = \frac{2 \pm \sqrt{4+16}}{2} = 1 \pm \sqrt{5}. \]

Thus the zeros of \(f(x)\) are −1, 1+\(\sqrt{5}\) and 1−\(\sqrt{5}\).

Finding that first zero
Quite often in my work as an applied mathematician, I come across an equation that I have to solve. That is, I need to find the zeros of some function. And sometimes what I need is a first zero, something that will let me start taking the function apart. But how do I find it? Do I make a guess? Am I lucky enough to find a small integer that works? Is the universe kind to me?

Well that's an interesting question. It turns out that the universe is kind. In fact, to mathematicians, it's incredibly, unexpectedly, awesomely kind. And that's because it's so "together," so connected, so coherent. You see the equation I have to solve isn't just an abstract mathematical equation—it comes out of a physical problem, it carries with it a rich context. And that context gives a physical meaning to the polynomial, and to all its various pieces, and that meaning often gives me a "hook" that let's me find a zero.

For example, last month I had to find the roots of the quartic equation

\[ x^4 - 2sx^3 + (2s^2-1)x^2 - s(s^2-1)x = 0. \]

This is actually a whole family of equations, one for every value of the parameter \(s\). And my job was to find the roots in terms of \(s\).

Well, \(x=0\) is clearly a root, so I can factor that out. But then what? I am left with a cubic equation. Well the context of the problem told me to expect that \(x=s\) would always be a root. So I could factor out \((x-s)\). And that left me a quadratic equation I could solve. (Problem 6).
Problems

1. Find all solutions of the following equations. The strategy here is to guess at a zero and then factor it out.
   (a) \( x^3 - 9x^2 - 12x + 20 = 0 \)
   (b) \( x^3 - 3x^2 - 2x + 4 = 0 \)

2. Write the polynomial \( f(x) = 5x - 4 \) in the form
   \( f(x) = (x-2)Q(x) + f(2) \)
for some polynomial \( Q(x) \).

3. Write the polynomial \( f(x) = x^4 - 11x^3 + 40x^2 - 55x + 25 \) in the form
   \( f(x) = (x-a)Q(x) + f(a) \)
for some polynomial \( Q(x) \) for the following values of \( a \).
   (a) \( a=0 \)
   (b) \( a=1 \)
   (c) \( a=2 \)
   (d) \( a=3 \)

4. At the right, we have drawn the graph of the polynomial
   \[ y = x^4 - 11x^3 + 40x^2 - 55x + 25. \]
   It intersects the x-axis at \( x=1 \) and \( x=5 \) and at two other places. Find these other two intersections.

5. Verify that if we set \( x = a \), we get a solution of the equation:
   \[ x^3 + (a-2)x^2 - 2a(1+a)x + 4a^2 = 0 \]
   Find two other solutions.

6. Verify that \( x=s \) is always a solution of the equation:
   \[ x^4 - 2sx^3 + (2s^2-1)x^2 - s(s^2-1)x = 0 \]
   and use this fact to find all the roots of the equation in terms of \( s \).
7. A feast of a problem. \{Notices of the AMS 47, March 2000, p. 360.\] Mark Saul a mathematician who teaches in the Bronxville school district was in Taiwan observing an after school class of 400 grade 10 students working on the following problem.

Find the remainder when the polynomial \( f(x) = x^{33} + x^{22} + x^{11} + x + 2 \) is divided by \( x^2 + x + 1 \).

Now of course if we had lots of time (and were kinda bored) we could do this by brute force long division, but the challenge is to find a clever way, and the factor theorem (or the ideas behind it) provide the lever. We know that if we did the long division we’d get an equation of the form:

\[
f(x) = (x^2 + x + 1)Q(x) + rx+s
\]

where the remainder \( rx+s \) will be a polynomial of degree 1. Now the factor theorem trick was to plug something into the equation which would make the \( Q(x) \) term zero. In the case of the factor theorem, the divisor was \( x-a \) and we could make it zero by setting \( x=a \). But here the divisor is quadratic, and so to make it zero we need to find one of its roots. If we had such a root \( \alpha \), then we could write:

\[
f(\alpha) = (\alpha^2 + \alpha + 1)Q(\alpha) + r\alpha + s = r\alpha + s
\]

where the last equality follows from the fact that \( \alpha \) was chosen to make \( \alpha^2 + \alpha + 1 = 0 \).

Now recall what the original problem is—to find the remainder \( rx+s \), and that means finding \( r \) and \( s \). Well what we have above is a single equation in \( r \) and \( s \). To solve for them we will need another equation (two unknowns usually requires two equations). Where should we get it from?

Well the divisor is quadratic, so it should have two roots, \( \alpha \) and a second root \( \beta \). Using them both, we would get two equations:

\[
\begin{align*}
  f(\alpha) &= r\alpha + s \\
f(\beta) &= r\beta + s
\end{align*}
\]

which we should be able to solve for \( r \) and \( s \).

Well that all sounds just fine, but the fly in the ointment (as they say) is that the divisor \( x^2 + x + 1 \) does not have any real roots. The quadratic formula gives us:

\[
x = \frac{-1 \pm \sqrt{1-4}}{2}
\]

and we get a negative number under the root sign.

But that’s not such a problem after all. One of the triumphs of mathematics is the discovery that we can handle such situations with standard algebra by simply treating \( \sqrt{-1} \) as a number whose square is \(-1\). Thus we write the two roots as

\[
\alpha = \frac{-1 \pm 3i}{2}, \quad \beta = \frac{-1 \pm 3i}{2}
\]

where \( i^2 = -1 \). So there we are.

Believe it or not there’s still a fascinating joker in the deck. When we come to calculate \( f(\alpha) \) and \( f(\beta) \) we will be faced with the problem of calculating numbers like \( \alpha^{33} \) and \( \beta^{33} \). How on earth do we do that? Well the divisor belongs to an interesting family. For any \( n \), the zeros of the polynomials \( x^n + x^{n-1} + \ldots + x + 1 \) have a wonderful algebraic property. Can you discover it? [Hint: factor \( x^n - 1 \).]
7. A little logic (goes a long way). Note that the converse of the Factor Theorem is immediately true. If I have a polynomial which has \((x-a)\) as a factor, then when I set \(x\) equal to \(a\) I will certainly get zero. And that also means the graph will cross the axis at \(x=a\).

But this fact does not give us a proof of the Factor Theorem. If we know that the converse of a theorem is true, that does not allow us to deduce that the theorem itself is true. Here we investigate that just a bit.

An implication is a statement of the form:

\[
\text{“if } A \text{ then } B,\]
\]

where \(A\) and \(B\) are statements which might or might not be true. The converse of the implication \(“\text{if } A \text{ then } B”\) is the implication

\[
\text{“if } B \text{ then } A.\]
\]

As an example, consider the implication:

1) If no one is at home, then the machine answers the phone.
   The converse of this is:
2) If the machine answers the phone, then no one is at home.

The implication \(“\text{if } A \text{ then } B”\) is said to hold if whenever \(A\) is true, \(B\) will also be true. For example, in a certain home the implication 1) above might well hold.

It is possible for an implication to hold but its converse to fail to hold. For example this might the case for 1) and 2) above—we might have 1) holding, but 2) not holding.

(a) Give an example of a mathematical implication that holds but whose converse fails to hold.
(b) Give an example of a mathematical implication that holds and whose converse holds.
(c) Give an example of a mathematical implication that fails to hold and whose converse fails to hold.

Often an implication is not stated in the nice \(“\text{if } A \text{ then } B”\) form, and sometimes care must be taken to see what it is really saying. Here are some examples.

3) Employees of the school board are not eligible.
4) You'll only find it on sale at the end of the season.
5) I never drink coffee before 10.
6) First come, first served.

Formulate each of these in standard form (if \(A\) then \(B\)) and then write the converse in as elegant as form as you can.

Note to avoid confusion. For some of the above questions there will be two correct answers, one implication being the contrapositive of the other. By definition, the contrapositive of the implication \(“\text{if } A \text{ then } B”\) is the implication \(“\text{if not-} A \text{ then not-} B.”\) For example, the contrapositive of 1) above is the implication 1*).

1*) If the machine does not answer the phone then someone is at home.
You can see that this really says the same thing as 1). We say that they are equivalent. In fact, an implication and its contrapositive are always equivalent.