Projective Monomial Curves

joint work with Dilip Patil

$$\mathcal{S} = \{m_0, \dots, m_{p+1}\}$$

Let m_i be integers, $0 < m_0 < m_1 < \cdots < m_{p+1}$, $\gcd(\{m_i\}) = 1$.
Γ = monoid generated by $\mathcal{S}$.
Θ_i = sums of i elements of $\mathcal{S}$ (repetitions allowed).

$$\mathfrak{M}_i := \Theta_i \setminus (\cup_{j<i}\Theta_j)$$

$\mathfrak{M}_i$ is the set of all integers that can be written as the sum of i elements of $\mathcal{S}$ but not fewer. $\mathfrak{M}_i$, is easily calculated recursively. In fact Θ_i is all sums of an element of $\mathcal{S}$ with an element of Θ_{i-1} and

$$\mathfrak{M}_i = (\mathfrak{M}_{i-1} + \mathcal{S}) \setminus (\cup_{j<i}\mathfrak{M}_j)$$

This has been programmed in Mathematica.
If $n \in \Gamma$ there is a unique integer i such that $n \in \mathfrak{M}_i$. Write $\mathrm{ord}_{\mathcal{S}}(n) = i$.
$\mathrm{gr}(\mathcal{S})_i$ = k-vector space with basis $\{t^n | n \in \mathfrak{M}_i\}$

$$\mathrm{gr}(\mathcal{S}) = \oplus_{i\geq 0}\mathrm{gr}(\mathcal{S})_i$$

We define a multiplication on $\mathrm{gr}(\mathcal{S})$ as follows:

$$t^a \cdot t^b := \begin{cases} t^{a+b}, & \text{if } \mathrm{ord}_{\mathcal{S}}(a+b) = \mathrm{ord}_{\mathcal{S}}(a) + \mathrm{ord}_{\mathcal{S}}(b), \\ 0, & \text{otherwise.} \end{cases}$$

With this multiplication $\mathrm{gr}(\mathcal{S})$ is a ring, and $H_{\mathrm{gr}(\mathcal{S})}(i) = \#\mathfrak{M}_i$.

Obviously $\mathrm{ord}_{\mathcal{S}}(a+b) \leq \mathrm{ord}_{\mathcal{S}}(a) + \mathrm{ord}_{\mathcal{S}}(b)$ so 'otherwise' will be $<$

Geometric interpretation.

Let $\varphi : S := k[X_0, \dots, X_{p+1}] \to k[t]$ be the k-algebra homomorphism defined by $X_i \mapsto t^{m_i}$, $I := \mathrm{Ker}(\varphi)$, let J the homogenisation of I with respect to a new variable W, and let $R := k[W, X_0, \dots, X_{p+1}]$. Then

$$\mathrm{gr}(\mathcal{S}) \cong R/(J, W)$$

R/J is the homogeneous co-ordinate ring of the projective closure C of the affine curve $S/I \cong$ the k-subalgebra of $k[t]$ generated by $\{t^n \mid n \in \Gamma\}$. Note that $R/J \cong k[u^{m_{p+1}}, t^{m_0}u^{m_{p+1}-m_0}, \ldots, t^{m_p}u^{m_{p+1}-m_p}, t^{m_{p+1}}]$ and C is the projective monomial curve with generic point

$$(u^{m_{p+1}}, t^{m_0}u^{m_{p+1}-m_0}, \ldots, t^{m_p}u^{m_{p+1}-m_p}, t^{m_{p+1}}).$$

The proof is just work your way through the definition of homogenization. More details in reference 3.

The Mathematica program is better than using Macaulay2 routines (e.g. for finding the Hilbert function) on the homogeneous co-ordinate ring R/J, in that Macaulay2 failed on some of our examples. It may be helpful that our algorithm is already at the difference sequence level. Groebner basis algorithms written specially for toric ideals should work better than general algorithms.

Under the isomorphism $R/J \cong k[u^{m_{p+1}}, t^{m_0}u^{m_{p+1}-m_0}, \ldots, t^{m_p}u^{m_{p+1}-m_p}, t^{m_{p+1}}]$ we can identify W with $u^{m_{p+1}}$. Furthermore $(u^{m_{p+1}}, t^{m_{p+1}})$ is a homogeneous system of parameters so R/J is Cohen-Macaulay if and only if $t^{m_{p+1}}$ is a non-zero-divisor in $R/(J, W) \cong \mathrm{gr}(S)$. (Recall that a graded ring is CM if and only if every homogeneous system of parameters is regular. W is of course a non-zero-divisor because R/J is an integral domain.)

Example

$\mathcal{S} = \{1, 2, 3\}$, $\Gamma = \mathbb{N}$

$\Theta_2 = \{2, 3, 4, 5, 6\}, \mathfrak{M}_2 = \{4, 5, 6\}$

$\Theta_3 = \{3, 4, 5, 6, 7, 8, 9\}, \mathfrak{M}_3 = \{7, 8, 9\}$.

Write the $\mathfrak{M}_i$ as rows

$$\begin{array}{c} 0 \\ 1\ 2\ 3 \\ 4\ 5\ 6 \\ 7\ 8\ 9 \end{array}$$

Here t^3 is a non-zero-divisor since $(1, 2, 3) + 3 = (4, 5, 6)$, $(4, 5, 6) + 3 = (7, 8, 9)$ etc. C is the twisted cubic.

The construction is an invariant of $\mathcal{S}$, not of Γ.

This method of writing the $\mathfrak{M}_i$ as rows will be used in later examples.

Another Example

$\mathcal{S} = \{1, 3, 4\}$. Then $\mathfrak{M}_i$ is row i (starting with $i = 0$) of

$$0$$

$$1\ 3\ 4$$

$$2\ 5\ 6\ 7\ 8$$

$$9\ 10\ 11\ 12$$

so $H_{\mathrm{gr}(\mathcal{S})} = 1\ 3\ 5\ 4\ \rightarrow$. Because $5 > 4$, t^4 cannot be a non-zero-divisor in $\mathrm{gr}(\mathcal{S})$, and $R/J \cong k[u^4, tu^3, t^3u, t^4]$ is not Cohen-Macaulay. The Hilbert function stabilizes at 4. (More precisely $\mathfrak{M}_{i+1} = 4 + \mathfrak{M}_i$ for $i \geq 3$)

This is everyone's favourite example of a non-Cohen-Macaulay curve.

Test for stopping

The curve C is of multiplicity m_{p+1} so $\#\mathfrak{M}_i = m_{p+1}$ and $t^{m_{p+1}} : \mathrm{gr}(\mathcal{S})_i \rightarrow \mathrm{gr}(\mathcal{S})_{i+1}$ is an isomorphism for i sufficiently large. Furthermore $\mathfrak{M}_i$ contains one element in each congruence class $\pmod{m_{p+1}}$. The following is an easy test for stopping our algorithm.

Theorem: If $t^{m_{p+1}} : \mathrm{gr}(\mathcal{S})_i \rightarrow \mathrm{gr}(\mathcal{S})_{i+1}$ is onto (equivalently every element of $\mathfrak{M}_{i+1}$ is the sum of m_{p+1} and an element of $\mathfrak{M}_i$) and $\#\mathfrak{M}_{i+1} = m_{p+1}$ then $t^{m_{p+1}} : \mathrm{gr}(\mathcal{S})_j \rightarrow \mathrm{gr}(\mathcal{S})_{j+1}$ is an isomorphism for $j \geq i + 1$. Equivalently adding $t_{m_{p+1}}$ gives a bijection from $\mathfrak{M}_j$ to $\mathfrak{M}_{j+1}$ for $j \geq i + 1$.

Proof: If $t^{m_{p+1}} : \mathrm{gr}(\mathcal{S})_i \rightarrow \mathrm{gr}(\mathcal{S})_{i+1}$ is onto then $(\mathrm{gr}(\mathcal{S})/t^{m_{p+1}}\mathrm{gr}(\mathcal{S}))_{i+1} = 0$. Since $\mathrm{gr}(\mathcal{S})/t^{m_{p+1}}\mathrm{gr}(\mathcal{S})$ is generated by elements of degree one we have $(\mathrm{gr}(\mathcal{S})/t^{m+1}\mathrm{gr}(\mathcal{S}))_j = 0$ for $j \geq i + 1$ from which it follows that $t^{m_{p+1}} : \mathrm{gr}(\mathcal{S})_j \rightarrow \mathrm{gr}(\mathcal{S})_{j+1}$ is onto for $j \geq i$. Since $\mathrm{gr}(\mathcal{S})_{i+1}$ has k-dimension m_{p+1} and $\dim_k \mathrm{gr}(\mathcal{S})_j = m_{p+1}$ for j sufficiently large the result follows.

Questions:

1. Does it suffice to have $t^{m_{p+1}}$ onto?
2. Does it suffice to have $\mathfrak{M}_{i+1}$ contain exactly one element of each congruence class $\pmod{m_{p+1}}$? (No!)

The general arithmetic sequence

Let $\mathcal{S} = \{m_0, m_1, \ldots, m_{p+1}\}$ where $m_i = m+id$, $0 \leq i \leq p+1$, $\gcd(m, d) = 1$. Then

$$\Theta_i = [im_0, im_{p+1}, d] := \{im_0 + jd,\ 0 \leq j \leq i(p+1)\}$$

$$\mathfrak{M}_i = \Theta_i \setminus \cup_{j\geq 1}\Theta_{i-jd} = [n, im_{p+1}, d]$$

where $n = \max(im_0, (i-d)m_{p+1} + d)$, so $\mathfrak{M}_i$ is either all of Θ_i or is the largest m_{p+1} elements of Θ_i. Furthermore $x \to x + m_{p+1}$ is an injection from $\mathfrak{M}_i \to \mathfrak{M}_{i+1}$ equivalently $t^{m_{p+1}} : \mathrm{gr}(\mathcal{S})_i \to \mathrm{gr}(\mathcal{S})_{i+1}$ is an inclusion, so R/J is Cohen-Macaulay.

The notation $[a, b, d]$ denotes the arithmetic sequence with differences d starting with a and ending with b.

A key idea here is that $\Theta_i \cap \Theta_j = \emptyset$ unless $i \equiv j \pmod{d}$. $\mathfrak{M}_i$ can contain no integer smaller than im_0 because im_0 is the smallest integer in Θ_i, and anything in the sequence smaller than $(i-d)m_{p+1} + d$ is in Θ_j for some $j < i$.

This argument is simpler than that in [1].

Example

$\mathcal{S} = \{3\ 5\ 7\}$

0

3 5 7

6 8 10 12 14

9 11 13 15 17 19 21

16 18 20 22 24 26 28

continuing with $\mathfrak{M}_{i+1}$ obtained from $\mathfrak{M}_i$ by adding 7. The sequence in $\mathfrak{M}_4$ continues downwards with 14, 12, but these are in $\mathfrak{M}_2$.

We define an element of $n \in \Gamma$ to be *unstable* if there exists $a \in \mathbb{N}$ such that

$$\mathrm{ord}_{\mathcal{S}}(am_{p+1} + n) < a + \mathrm{ord}_{\mathcal{S}}(n)$$

If no such a exists then we say that n is stable. (Equivalently n is unstable if and only if the element $t^n \in \mathrm{gr}(\mathcal{S})$ is killed by some power of $t^{m_{p+1}}$). Then we have that $\mathrm{gr}(\mathcal{S})$ is Cohen-Macaulay if and only if $\mathcal{S}$ has no unstable elements.

A more complicated example.

Let $\mathcal{S} = \{d, m, m_1, \ldots, m_{p+1}\}$, where $m_i = m + id$ $(0 \leq i \leq p+1)$, $m > 1$, $\gcd(d, m) = 1$, $p \geq 0$. This is an example of an almost arithmetic progression, much studied by Dilip Patil and others. It was Patil's idea to consider this set, as the general almost arithmetic progression seems complicated.

There is not a unique way of representing an element of Γ as the sum of elements of $\mathcal{S}$. We found the following representation convenient:

$$n = id + cm_{p+1} \text{ with } 0 \leq i \leq m_{p+1} - 1, c \in \mathbb{Z}$$

even though a finite number of $n \in \Gamma$ will require $c < 0$.

Enumeration of cases:

1. If $0 \leq i \leq d < m$ and $c < 0$ then $n \notin \Gamma$
2. If $n = id + cm_{p+1}$ with $c \geq 0$ and $0 \leq i \leq d$ then $\mathrm{ord}_{\mathcal{S}}(n) = i + c$ and n is a stable element of Γ.
3. Suppose $d > m$ and $n = id + cm_{p+1}$ with $c < 0, 0 \leq i < d$. Then $n \in \Gamma$ if and only if $m - (p+1)c \leq i < d$. Furthermore $\mathrm{ord}_{\mathcal{S}}(n) = c + d$, and n is an unstable element of Γ.
4. Suppose $n = id + cm_{p+1}$ with $d \leq i < m_{p+1}$ and $n \geq (c+d)m$. Then n is a stable element of Γ and $\mathrm{ord}_{\mathcal{S}}(n) = d + c$.
5. Suppose $n = id + cm_{p+1}$ with $c < 0$ and $n < (c+d)m$. Then $n \notin \Gamma$.
6. Suppose $n = id + cm_{p+1}$ with $c \geq 0$, $d < i < m_{p+1}$ and $n < (c+d)m$. Then n is an unstable element of Γ and $\mathrm{ord}_{\mathcal{S}}(n) = i + c$ (necessarily $d < m$).

The Hilbert function H of $\mathrm{gr}(\mathcal{S})$ can be found by counting the number of integers of a given order. Also there are no unstable integers if and only if $d = m - 1$ or $m < d \leq m + p + 1$, so these are the Cohen-Macaulay cases.

The case $d > m$ is the simplest:

Theorem 1. Suppose that $d > m$. Then $H(j) = 1 + (p+2)j$ for $0 \leq j \leq d-1$ and $H(j) = m_{p+1}$ for $j \geq d$.

The case $d < m$ is more difficult:

I state first a couple of the easier cases of $d < m$, even though they can be obtained from the general case (Theorem 4) below.
Theorem 2. Let $d < m$ and $p = 0$. Then $H(j) = 1 + 2j$, $0 \leq j \leq m-1$, and $H(j) = m + d$ $(= m_{p+1})$ for $j \geq m$.
Theorem 3. Let $d = m-1$ and $p > 0$. Then $H(j) = 1+(p+2)j$, $0 \leq j \leq d$ and $H(j) = m_{p+1}$ for $j \geq d$.
Notation:

$$\beta = \left\lfloor \frac{m_{p+1} - d}{p+1} \right\rfloor = \left\lfloor \frac{m + dp}{p+1} \right\rfloor = \left\lfloor \frac{m_p}{p+1} \right\rfloor$$

$$\sigma = \left\lceil \frac{m-d-1}{p+1} \right\rceil, \ \tau = m - d - 1 - (p+1)(\sigma - 1)$$

The number of stable elements of case 4 of order i is $i(p+1)$ as long as $i(p+1) \leq m_p$, then stabilizes at value m_p. β is the largest order in which the formula $i(p+1)$ can hold. The other parameters are explained in reference [2].
Theorem 4. Let $d < m-1$ and $p \geq 1$. Then

1. If $m + pd \equiv 0 \pmod{p+1}$ then $\Delta H(0) = 1$, $\Delta H(j) = p+2$, $1 \leq j \leq \beta - 1$, $\Delta H(\beta) = p+1$, and $\Delta H(j) = -1$ for $j = \beta + \tau + p\ell$, $0 \leq \ell < \sigma$.

2. If $m+pd \equiv 1 \pmod{p+1}$ then $\Delta H(0) = 1$, $\Delta H(j) = p+2$, $1 \leq j \leq \beta$, and $\Delta H(j) = -1$ for $j = \beta + \tau + p\ell$, $0 \leq \ell < \sigma$.

3. If $m + pd \equiv \ell \pmod{p+1}$, $2 \leq \ell < p+1$, then $\Delta H(0) = 1, \Delta H(j) = p+2$, $1 \leq j \leq \beta$, $\Delta H(\beta+1) = \ell$, and $\Delta H(j) = -1$ for $j = \beta + 1 + \tau + p\ell$, $0 \leq \ell < \sigma$.

In each case $\Delta H(j) = 0$ for all other values of j.

Example 1:

Let $\mathcal{S} = \{4, 11, 15, 19\}$, so that $d = 4$, $m = 11$, $p = 1$. The following table gives the sets $\mathfrak{M}_j$, $1 \leq j \leq 11$. Subsequent rows are obtained by adding $19 = m_{p+1}$ to the preceding row.

$$\{4,11,15,19\}$$
$$\{8, 22, 23, 26, 30, 34, 38\}$$
$$\{12, 27, 33, 37, 41, 42, 45, 49, 53, 57\}$$
$$\{16, 31, 44, 46, 48, 52, 56, 60, 61, 64, 68, 72, 76\}$$
$$\{20, 35, 50, 55, 59, 63, 65, 67, 71, 75, 79, 80, 83, 87, 91, 95\}$$
$$\{24, 39, 54, 66, 69, 70, 74, 78, 82, 84, 86, 90, 94, 98, 99, 102, 106, 110, 114\}$$
$$\{28, 43, 58, 73, 77, 81, 85, 88, 89, 93, 97, 101, 103, 105, 109, 113, 117, 118, 121, 125, 129, 133\}$$
$$\{32, 47, 62, 92, 96, 100, 104, 107, 108, 112, 116, 120, 122, 124, 128, 132, 136, 137, 140, 144, 148, 152\}$$
$$\{36, 51, 111, 115, 119, 123, 126, 127, 131, 135, 139, 141, 143, 147, 151, 155, 156, 159, 163, 167, 171\}$$
$$\{40, 130, 134, 138, 142, 145, 146, 150, 154, 158, 160, 162, 166, 170, 174, 175, 178, 182, 186, 190\}$$
$$\{149, 153, 157, 161, 164, 165, 169, 173, 177, 179, 181, 185, 189, 193, 194, 197, 201, 205, 209\}$$

In the table the black entries are stable by case 4. These have order $c+d$. For a given order, c is fixed and i varies, so the integers $n = id + cm_{p+1}$ of that order are the consecutive elements of an arithmetic progression with differences d. The largest integer m_{p+1} extends the progression but was not counted as part of case 4 in our enumeration of cases.

The magenta entries are stable by case 2. They have order $i + c$ so for fixed order, c increases by one as i decreases by one, so the integers n of this order are the consecutive terms of an arithmetic progression with differences $m_{p+1} - d = m_p$ (here equal to 15).

The green are covered by both cases 2 and 4.

The red entries are unstable by case 6. In general (for $d < m$) they form the array

$$\begin{array}{lllll}
(d+1)d & (d+2)d & \dots & & (m-1)d \\
(d+1)d + m_{p+1} & (d+2)d + m_{p+1} & \dots & (m-1-(p+1))d + m_{p+1} & \\
 & & & \vdots &
\end{array}$$

which here is

20	24	28	32	36	40
39	43	47	51		
58	62				

Elements of a given order are the upwards diagonals starting at the upper left hand corner. Initially these elements extend the case 2 (magenta) progression downwards, but then there are fewer of them, leading to the complexity of Theorem 4.

Example 2:

Let $\mathcal{S} = \{8, 3, 11, 19\}$, so that $d = 8$, $m = 3$, $p = 1$. The following table gives the sets $\mathfrak{M}_j$, $1 \leq j \leq 8$. Subsequent rows are obtained by adding $19 = m_{p+1}$ to the preceding row.

{3,8,11,19}
{6,14,16,22,27,30,38}
{9,17,24,25,33,35,41,46,49,57}
{12,20,28,32,36,43,44,52,54,60,65,68,76}
{15,23,31,39,40,47,51,55,62,63,71,73,79,84,87,95}
{18,26,34,42,48,50,58,59,66,70,74,81,82,90,92,98,103,106,114}
{21,29,37,45,53,56,61,67,69,77,78,85,89,93,100,101,109,111,117,122,125,133}
{64,72,75,80,86,88,96,97,104,108,112,119,120,128,130,136,141,144,152}

The black entries are again stable by case 4. As in the previous example they form an arithmetic sequence with differences d which here is equal to 8.

The magenta entries are again stable by case 2. As in the previous example they form an arithmetic sequence with differences m_{p+1} which here is 11.

The red entries are unstable by case 3, and continue the case 4 progression, leading to a deceptively simple Theorem 1.

Now suppose that $\mathrm{gr}(\mathcal{S})$ is Cohen-Macaulay.

Definition Let $\mathcal{S} = \{n_0, n_1, \ldots, n_{p+1}\}$ with $1 \leq n_0 < n_1 < \cdots < n_{p+1}$ and suppose that $t^{n_{p+1}}$ is a non-zero-divisor in $\mathrm{gr}(\mathcal{S})$. Let $\mathcal{S}' = \{n_0, n_1, \ldots, n_p\}$ and Γ the submonoid of $\mathbb{N}$ generated by $\mathcal{S}$. Let $x \in \Gamma \backslash (n_{p+1} + \Gamma)$. Then we will say that x is a maximal element of $\Gamma \backslash (n_{p+1} + \Gamma)$ if for all $j \in \mathcal{S}'$ we have either $x + j \in m_{p+1} + \Gamma$ or $\mathrm{ord}_{\mathcal{S}}(x + j) < \mathrm{ord}_{\mathcal{S}}(x) + \mathrm{ord}_{\mathcal{S}}(j)$.

The definition has been set up so that x is maximal if and only if t^x is a non-zero element of $\mathrm{gr}(\mathcal{S})/t^{n_{p+1}}\mathrm{gr}(\mathcal{S})$ which annihilates the homogeneous maximal ideal of $\mathrm{gr}(\mathcal{S})/t^{n_{p+1}}\mathrm{gr}(\mathcal{S})$. Since $\Gamma \backslash (n_{p+1} + \Gamma)$ is finite the maximal elements are readily determined and the Cohen-Macaulay type of R/J is the number of maximal elements.

The following result gives the type of our Cohen-Macaulay examples.

Theorem Let $\mathcal{S} = \{d, m_0, \ldots, m_{p+1}\}$ where $\gcd(m, d) = 1$, $m_i = m + id$ for $0 \le i \le p+1$, and Γ the monoid generated by $\mathcal{S}$. Then the k-algebra $\mathrm{gr}(\mathcal{S})/t^{m_{p+1}}\mathrm{gr}(\mathcal{S})$ is level, and the Cohen-Macaulay type of $\mathrm{gr}(\mathcal{S})$ is $p+2$ if $d = m-1$, $m_{p+1} - 1 - (p+2)(d-1)$ if $m < d < m+p+1$, and $p+2$ if $d = m+p+1$.

Example Let $\mathcal{S} = \{5, 4, 9, 14, 19\}$, so that $d = 5$, $m = 4$ and $p = 2$. The j^{th} row of the following table is $\mathfrak{M}_j$, $j \ge 1$. The black entries in row j are $19 + \mathfrak{M}_{j-1}$. The remaining entries $\mathfrak{N}_j$ in row j yield a basis of $(\mathrm{gr}(\mathcal{S})/t^{m_{p+1}}\mathrm{gr}(\mathcal{S}))_j$.

$$\{4, 5, 9, 14, 19\}$$
$$\{8, 10, 13, 18, 23, 24, 28, 33, 38\}$$
$$\{12, 15, 17, 22, 27, 29, 32, 37, 42, 43, 47, 52, 57\}$$
$$\{16, 20, 21, 26, 31, 34, 36, 41, 46, 48, 51, 56, 61, 62, 66, 71, 76\}$$
$$\{25, 30, 35, 39, 40, 45, 50, 53, 55, 60, 65, 67, 70, 75, 80, 81, 85, 90, 95\}$$
$$\{44, 49, 54, 58, 59, 64, 69, 72, 74, 79, 84, 86, 89, 94, 99, 100, 104, 109, 114\}$$

For example $\mathfrak{N}_5 = \{25,\ 30\}$ and $20 + 5 = 25$, $16 + 9 = 25$, $21 + 9 = 30$, $26 + 4 = 30$ so none of the elements in $\mathfrak{N}_4$ is maximal. Similarly no elements of $\mathfrak{N}_j$ are maximal for $j < 4$. The algebra $\mathrm{gr}(\mathcal{S})/t^{m_{p+1}}\mathrm{gr}(\mathcal{S})$ is level, with its socle having k-basis t^{25}, t^{30}, both of order 5, and the type is $2 = 19 - 1 - 4 \cdot 4$, as claimed.

Let $\mathcal{S}' = \{1, m, m+1, \ldots, m+p+1\}$ (so this is the previous type of example where $d = 1$, and m and p have their usual meanings), and let $\mathcal{S} = \{1, m, m+1, \ldots, m+p+1, n\}$ for some $n > m+p+1$. Let Γ be the submonoid of $\mathbb{N}$ generated by $\mathcal{S}$ and Γ' the submonoid generated by $\mathcal{S}'$. (Since $d = 1$, $\Gamma = \Gamma' = \mathbb{N}$ but elements of $\mathbb{N}$ will have different orders as elements of Γ or Γ'). We determine a necessary and sufficient for $\mathcal{S}$ to be Cohen-Macaulay, and find the Cohen- Macaulay type of those that are Cohen-Macaulay.

Theorem Let $\mathcal{S}'$ and $\mathcal{S}$ be as defined above. Then $\mathrm{gr}(\mathcal{S})$ is Cohen-Macaulay if and only if $n > (m-1)(m+p)$.

Theorem Let $\mathcal{S}'$ and $\mathcal{S}$ be as above, with $n > (m-1)(m+p)$. Then the Cohen-Macaulay type of $\mathrm{gr}(\mathcal{S})$ is $\sigma + \ell$ where $\sigma = \lceil \frac{m-2}{p+1} \rceil$ and

$$\ell = \begin{cases} m+p+1, & \text{if } n \equiv 1 \pmod{m+p+1}; \\ m-1, & \text{if } n \equiv i \pmod{m+p+1} \text{ where } 2 \le i \le m; \\ i-1, & \text{if } n \equiv i \pmod{m+p+1} \text{ where } m < i \le m+p+1. \end{cases}$$

Example:

Let $\mathcal{S}' = \{1, 7, 8, 9\}$ so that $m = 7,\ p = 1$. Then $(m-1)(m+p)$= 48 so for $\mathcal{S} = \{1, 7, 8, 9, n\}$ then $\mathrm{gr}(\mathcal{S})$ is Cohen-Macaulay if and only if $n \geq 49$. Since $d = 1$, $[0, 48] = \Gamma\backslash(49 + \Gamma)$ and the orders relative to Γ of integers in $[0, 48]$ are the same as for Γ':

$$\begin{gathered}\{0\}\\ \{1,\ 7,\ 8,\ 9\}\\ \{2,\ 10,\ 14,\ 15,\ 16,\ 17,\ 18\}\\ \{3,\ 11,\ 19,\ 21,\ 22,\ 23,\ 24,\ 25,\ 26,\ 27\}\\ \{4,\ 12,\ 20,\ 28,\ 29,\ 30,\ 31,\ 32,\ 33,\ 34,\ 35,\ 36\}\\ \{5,\ 13,\ 37,\ 38,\ 39,\ 40,\ 41,\ 42,\ 43,\ 44,\ 45\}\\ \{6,\ 46,\ 47,\ 48\}\end{gathered}$$

Here the unstable elements are $\{2\}$, $\{3\ 11\}$, $\{4\ 12\ 20\}$, $\{5\ 13\}$ $\{6\}$ (grouped by order) and $\sigma = 3$ with the unstable maximal elements being 6 13 20, and 46 covers 37 and 45, 47 covers 38, 48 covers 39, 40, 41. The stable elements of order 5 that are not covered by stable elements of order 6 are 42, 43, 44, so the Cohen-Macaulay type of $\mathrm{gr}(\mathcal{S})$ is 9, and the socle of $\mathrm{gr}(\mathcal{S})/t^{49}\mathrm{gr}(\mathcal{S})$ has basis t^6, t^{13}, t^{20}, t^{42}, t^{43}, t^{44}, t^{46}, t^{47}, t^{48}.

Here $n = 49$, $m + p + 1 = 7 + 1 + 1 = 9$, $n \equiv 4 \pmod 9$ so $\ell = m - 1 = 6$ and $\sigma + \ell = 3 + 6 = 9$ as we have obtained.

References:

1. S. Molinelli and G. Tamone On the Hilbert function of certain rings of monomial curves, JPAA,101:191-206,1995.

2. D.P.Patil and L.Roberts Hilbert functions of monomial curves (in preparation)

3. L.G.Roberts Certain projective curves with unusual Hilbert function, JPAA,104:303-311,1995.

Questions:

1. In all cases proved here and and all computed examples of the Hilbert function of a projective monomial curve, the h-vector has no negative entries if and only if the homogeneous co-ordinate ring is Cohen-Macaulay. Is this true in general? (The 'if' direction is of course trivial.)

2. Is the h-vector always unimodal in the Cohen-Macaulay case?