MONOMIAL SUBRINGS IN ARBITRARY DIMENSION

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ABSTRACT. We study the seminormalization and weak normalization of subrings of $\mathbb{Z}[t_1, \ldots, t_n]$ generated by monomials. This generalizes our earlier results with $n = 1$.

1. INTRODUCTION.

We wish to consider subrings A of $\mathbb{Z}[t_1, \ldots, t_n]$ generated by terms $x_{\mathbf{M}}\mathbf{t}^{\mathbf{M}}$, $x_{\mathbf{M}} \in \mathbb{Z}$, $\mathbf{M} = (m_1, \ldots, m_n)$, where as usual $\mathbf{t}^{\mathbf{M}}$ means $t_1^{m_1} t_2^{m_2} \ldots t_n^{m_n}$. (If $x_{\mathbf{M}} = 1$ the term will be called a monomial.) Such rings A will be called monomial subrings of $\mathbb{Z}[t_1, \ldots, t_n]$. Monomial subrings of $\mathbb{Z}[t]$ arose naturally from our study of weakly subintegral ring extensions, for example in [13, Section 3]. In [12] we determined the seminormal and weakly normal monomial subrings of $\mathbb{Z}[t]$ which satisfied the additional condition that $a_i = 1$ for some $i > 0$. Our goals in this paper are to generalize these results to the case $n > 1$ (which we do in Theorems 4.10, 4.11). This requires us to use monoids and the geometry of polytopes in a more systematic way than in the $n = 1$ case. We also attempt to drop the requirement that some $a_i = 1$, obtaining our most explicit results if $n = 1$ (Theorems 6.1, 6.2). In Section 5 we give examples of our various constructions. We also an example of the conductor of a submonoid of $\mathbb{N}^n$ in its normalization (Theorem 5.2), which generalizes to $n > 1$ the simplest case of the postage stamp problem. This result is perhaps of independent interest. We hope to study more general rings constructed in a similar manner. If $\mathbb{Z}$ is replaced by a principal ideal domain we expect most, if not all, results to go through with minor modification. However there are many other possibilities that we hope to address in the future.

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Throughout, $\mathbb{Z}_+$ is the set of positive integers $\{1, 2, 3, \ldots\}$ and $\mathbb{N}$ is the set of non-negative integers $\{0, 1, 2, 3, \ldots\}$. (Caution: several of our earlier papers have reversed this notation! However here we use $\{0, 1, 2, 3, \ldots\}$ the most, so it seems more convenient to use the simpler notation for the latter.) The non-negative real and rational numbers will be denoted by $\mathbb{R}_{\geq 0}$ and $\mathbb{Q}_{\geq 0}$ respectively, and the positive reals and rationals will be denoted by $\mathbb{R}_+$ and $\mathbb{Q}_+$. Polytopes, polyhedra, and cones will be as defined in [19]. In particular a polytope is the convex hull of a finite set of points in $\mathbb{R}^n$, and a cone is the set of all linear combinations with coefficients in $\mathbb{R}_{\geq 0}$ of a (not necessarily finite) set of points in $\mathbb{R}^n$. Cones are convex and contain $\mathbf{0} \in \mathbb{R}^n$.

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Definition-Notation 1.1. The support of the monomial subring A is $\mathcal{S} := \{\mathbf{M} \in \mathbb{N}^n \mid x_{\mathbf{M}}\mathbf{t}^{\mathbf{M}} \in A$ for some $x_{\mathbf{M}} \in \mathbb{Z}, x_{\mathbf{M}} \neq 0\}$. If $\mathbf{M} \in \mathcal{S}$ let $a_{\mathbf{M}}$ be the smallest positive integer such that $a_{\mathbf{M}}\mathbf{t}^{\mathbf{M}} \in A$. If $\mathbf{M} \notin \mathcal{S}$ let $a_{\mathbf{M}} = 0$. We will refer to $a_{\mathbf{M}}$ as the coefficient of $\mathbf{t}^{\mathbf{M}}$ in A.

Clearly $a_{\mathbf{0}} = 1$ and $\mathcal{S}$ is a submonoid of $\mathbb{N}^n$ (i.e. a subsemigroup containing $\mathbf{0}$). We regard A as graded by $\mathcal{S}$ or by $\mathbb{N}^n$. The homogeneous elements for this grading are just the terms, and A has a homogeneous $\mathbb{Z}$-basis $\{a_{\mathbf{M}}\mathbf{t}^{\mathbf{M}}\}_{\mathbf{M}\in\mathcal{S}}$. We will assume that $\mathcal{S} \neq \{\mathbf{0}\}$, i.e. that A is more than just $\mathbb{Z}$. We have that $A \subseteq \mathbb{Z}[\mathcal{S}]$, the latter being the monoid ring of $\mathcal{S}$.

Notation 1.2. If $\mathcal{T}$ is a submonoid of $\mathbb{N}^n$, let $\mathbb{Z}\mathcal{T}$ be the subgroup of $\mathbb{Z}^n$ generated by $\mathcal{T}$.

If the expression for the monoid is complicated, we will use parentheses, e.g. $\mathbb{Z}(\mathcal{S} \cap \mathcal{T})$, for the subgroup generated by $\mathcal{S} \cap \mathcal{T}$.

Lemma 1.3. *Let $\{a_{\mathbf{M}}\}_{\mathbf{M}\in\mathbb{N}^n}$ be an arbitrary collection of non-negative integers. Then $\oplus a_{\mathbf{M}}\mathbf{t}^{\mathbf{M}}$ is a ring if and only if*

$$\mathcal{S} = \{\mathbf{M} \mid a_{\mathbf{M}} \neq \mathbf{0}\} \text{ is a submonoid of } \mathbb{N}^n \tag{1.3.1}$$

$$a_{\mathbf{0}} = 1 \tag{1.3.2}$$

$$\text{If } \mathbf{M}_1, \mathbf{M}_2 \in \mathcal{S}, \text{ then } a_{\mathbf{M}_1+\mathbf{M}_2} \mid a_{\mathbf{M}_1}a_{\mathbf{M}_2}. \tag{1.3.3}$$

Recall that a submonoid of $\mathbb{N}^n$ is a subset $\mathcal{S}$ such that $\mathbf{0} \in \mathcal{S}$ and $(\mathbf{M}_1, \mathbf{M}_2 \in \mathcal{S} \Rightarrow \mathbf{M}_1 + \mathbf{M}_2 \in \mathcal{S})$. It is not hard to satisfy these conditions, so there are lots of such rings. Condition 1.3.3 can be checked at each prime, so it suffices to consider rings where all the $a_{\mathbf{M}}$ are powers of one prime, and then put the rings together by multiplying the coefficients degreewise. If $a \in \mathbb{Z}_+$ and q is a prime, denote by $a^{[q]}$ the q-primary part of a (e.g. $12^{[3]} = 3$ and $12^{[2]} = 2^2 = 4$). By convention $0^{[q]} = 0$. More formally we have the following:

Lemma 1.4. *If $\{a_{\mathbf{M}}\}_{\mathbf{M}\in\mathbb{N}^n}$ is an arbitrary collection of non-negative integers, $A = \oplus a_{\mathbf{M}}\mathbf{t}^{\mathbf{M}}$, and $A^{[q]} = \oplus a^{[q]}_{\mathbf{M}}\mathbf{t}^{\mathbf{M}}$ then A is a ring if and only if $A^{[q]}$ is a ring for all primes q. Furthermore $A = \cap A^{[q]}$.* □

Note that $A^{[q]}$ is the set of all elements x of $\mathbb{Z}[t_1, \ldots, t_n]$ (or of $\mathbb{Z}[\mathcal{S}]$) such that $dx \in A$ for some integer d relatively prime to q.

2. Examples if $n = 1$.

The possibilities for conditions 1.3.$*$ are already quite rich for $n = 1$, in which case $\mathcal{S} \subseteq \mathbb{N}$. For example the sequence $\{a_i\}_{i\in\mathbb{N}}$ (with $a_0 = 1$) is a solution to (1.3.3) if $a_{i+1} \mid a_i$ for $i \geq 1$, or if $a_i = d^i$ for some integer d, and any two solution sequences can be multiplied together termwise to yield another solution. (Since the index is in $\mathbb{N}$ it seems more natural to write a_i instead of $a_{\mathbf{M}}$). Without loss of generality we can assume that $\mathrm{GCD}(\{i \mid i \in \mathcal{S}\}) = 1$ (i.e. all sufficiently large integers are in $\mathcal{S}$). Suppose in addition that $a_i = 1$ for some $i \in \mathcal{S}, i > 1$. This is the case that arose first in [12] and will hopefully lead to more tractable examples. Let $d = \mathrm{GCD}(\{i \in \mathcal{S} \mid a_i = 1\})$. We observed in [12] that eventually the sequence $\{a_i\}$ becomes periodic mod d. Furthermore the resulting stabilization sequence also satisfies the conditions 1.3.$*$. Hence it is of interest to consider sequences $\{a_i \mid a_i \in \mathbb{Z}_+\}_{i\in\mathbb{N}}$ which satisfy (1.3.$*$) and are periodic mod d, i.e. $a_{i+d} = a_i$ for all

$i \in \mathbb{N}$. Here $\mathcal{S} = \mathbb{N}$. By the discussion of the preceding paragraph we can replace a_i by the exponent e_i (≥ 0) to which some prime q appears in a_i. Then (1.3.3) becomes

$$e_{i+j} \leq e_i + e_j \ (i, j \in \mathbb{Z}/d\mathbb{Z}, \text{ the sum } i + j \text{ also taken in } \mathbb{Z}/d\mathbb{Z}) \tag{2.1}$$

The conditions $e_i \geq 0$, which we also want, are a consequence of (2.1). For (2.1) implies $e_{(d+1)i} \leq (d+1)e_i$, which is obviously impossible if $e_i < 0$ since $(d+1)i = i$ (in $\mathbb{Z}/d\mathbb{Z}$, where the subscripts live). We have $e_0 = 0$, and if i, j, or $i + j$ is equal to 0 the inequalities (2.1) are trivially satisfied. Hence we need consider only those inequalities (2.1) in which none of i, j, or $i + j$ is equal to 0. The solutions to (2.1) in $\mathbb{R}^{d-1}_{\geq 0}$ form a cone C_d and we are interested in the integer points in this cone.

Examples 2.2. i) If $d = 2$ then we have only e_1 which can be arbitrary, since there are no conditions in this case.

ii) If $d = 3$ then we have only e_1 and e_2 with inequalities $e_2 \leq 2e_1$ and $e_1 \leq 2e_2$. Thus e_1 can be an arbitrary non-negative integer and e_2 can be any integer satisfying $e_1/2 \leq e_2 \leq 2e_1$.

iii) Now consider $d = 4$. Here the inequalities are

$$e_2 \leq 2e_1$$

$$e_3 \leq e_1 + e_2$$

$$e_1 \leq e_2 + e_3$$

$$e_2 \leq 2e_3$$

The solutions to these inequalities are the points (a, c, b) ($c \in \mathbb{N}, a \geq c/2$, $\mathrm{Max}(a - c, c/2) \leq b \leq a + c$).

Geometric representations of C_d become increasingly difficult as d increases. One of the standard ways of representing a (finitely generated rational) cone in $\mathbb{R}^{d-1}_{\geq 0}$, such as C_d, is as the non-negative span $\{\Sigma \lambda_i \mathbf{r}_i \mid \lambda_i \in \mathbb{R}_{\geq 0}\}$ of a finite set of elements $\mathbf{r}_i \in \mathbb{Q}^{d-1}_{\geq 0}$ called rays [19, Theorem 1.2, page 30]. The $\mathbf{r}_i$ are unique up to positive rational multiple if chosen to be irredundant, and can be taken to have integer coefficients that have no common factor. One can then intersect C_d with a hyperplane in $\mathbb{R}^{d-1}$, say $e_1 + e_2 + \ldots + e_{d-1} = 1$. This yields a polytope $\widehat{C}_d$ in $\mathbb{R}^{d-2}$, so visualization becomes possible for larger d. The rays intersect the hyperplane in the vertices of $\widehat{C}_d$.

If $d = 4$ it follows from the discussion above that there are four rays ($\{1, 2, 1\}$, $\{3, 2, 1\}, \{1, 2, 3\}, \{1, 0, 1\}$) The cross-section $\widehat{C}_4$ of C_4 by the plane $e_1 + e_2 + e_3 = 1$ is thus the kite-shaped shaded region, illustrated in Figure 1 using barycentric co-ordinates (e_1, e_2, e_3).

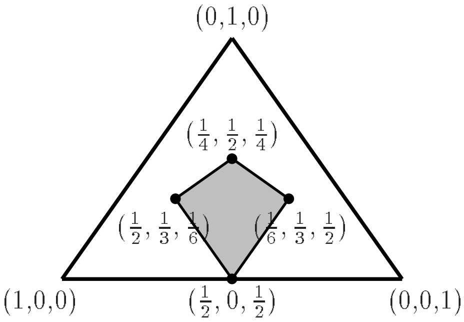


Figure 1.

If $d = 5$ there are 8 rays, and the cross-section $\widehat{C}_5$ is a polytope in $\mathbb{R}^3$. It is two tents placed base to base, rotated 90°, as illustrated in Figure 2. In the figure, visible edges are solid and hidden edges are dashed. The common base of the tents is a square, and the two sides of each tent are equilateral trapezoids with the top edge two-thirds of the length of the base. The triangular ends of the tent are isosceles, with the angle at the peak approximately 84°. The four trapezoidal faces are congruent and the four triangular faces are congruent.

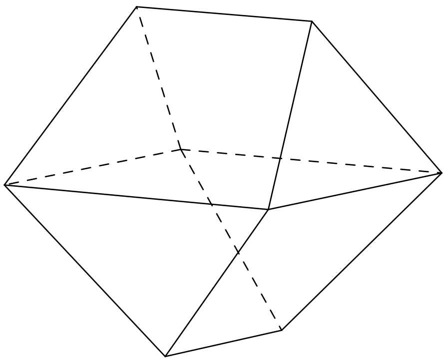


Figure 2.

Even $d = 5$ is beyond convenient hand calculation. We found the rays using several of the computer programs mentioned below.

Lemma 2.3. *The algebra $A = \mathbb{Z}[\{a_it^i\}_{i\geq 1}]$ has the following presentation as a $\mathbb{Z}$-algebra: Let u_i be indeterminates over $\mathbb{Z}$ and map u_i to a_it^i. Then $A \cong \mathbb{Z}[\{u_i\}_{i\geq 1}]/(\{u_iu_j - (a_ia_j/a_{i+j})u_{i+j}\}_{i\geq 1, j\geq 1})$. If $a_d = 1$ and the sequence a_i is periodic of period d then A has generators $\{u_i\}_{1\leq i\leq d}$, and relations $\{u_iu_j - (a_ia_j/a_{<i+j>})u_d^{\epsilon}u_{<i+j>}\}_{1\leq i\leq d, 1\leq j\leq d}$, where $< i+j >$ denotes the representative of $i+j$ mod d in the interval $[1, d]$ and $\epsilon_d = 0$ if $i+j \leq d$ and $\epsilon_d = 1$ if $i+j > d$.*

□

We believe that rings corresponding to different points in C_d are non-isomorphic. One example of this will be illustrated here, using the presentation in Lemma 2.3. Take the prime q to be 2, and let $A = \mathbb{Z}[2t, 2t^2, \dots, 2t^{d-1}, t^d]$ corresponding to $(1, 1, \dots, 1) \in C_d$, and $B = \mathbb{Z}[4t, 4t^2, \dots, 4t^{d-1}, t^d]$ corresponding to $(2, 2, \dots, 2) \in C_d$. If $A \cong B$ (as rings) then $A/4A \cong B/4\mathrm{B}$. From the presentation above it is easily seen that the nil radicals $\mathrm{nil}(A/4A)$ and $\mathrm{nil}(B/4B)$ are both generated by (the canonical images of) $u_1, \dots, u_{d-1}$ and that $\mathrm{nil}(B/4B)^2 = 0$ but $\mathrm{nil}(A/4A)^2 \neq 0$ which is a contradiction. Therefore $A \ncong B$.

Let us now establish some facts about the inequalities 2.1.

Lemma 2.4. *The system (2.1) is equivalent to the inequalities*

$$e_{i+j} \leq e_i + e_j \ (i, j \in \mathbb{Z}/d\mathbb{Z},\ \textit{none of } i, j, i+j \textit{ equal to } 0).$$

There are $\binom{d}{2} - \lceil (d-1)/2 \rceil$ *such inequalities, none of which can be omitted.*

Proof. There are $\binom{d}{2}$ ways of choosing i, j from $[1, \dots, d-1]$ with repetition allowed. $\lceil (d-1)/2 \rceil$ is the number of these choices whose sum is d, and hence are no longer needed. To see that none of the remaining inequalities can be omitted note that $e_{i+j} = 3, e_i = 1, e_j = 1, e_k = 2$ for $k \neq i, j, i{+}j$ satisfies all inequalities except $e_{i+j} \leq e_i + e_j$, so the latter cannot be omitted. (In this argument $i = j$ is allowed). □

Remark 2.5. There are several computer programs which will find the rays starting from the hyperplanes in Lemma 2.4 ([6], [2], [4]). We explored the cones C_d $(4 \leq d \leq 13)$ with some of these programs, working mostly with [6] (which is the first one that we found out about). The cones are quite complicated. Detailed observations will appear elsewhere. Here we will content ourselves with the following table, which lists the number of hyperplanes as determined by Lemma 2.4 and the number of rays, as determined by the computer programs:

Table 2.6.

d	hyperplanes	rays
4	4	4
5	8	8
6	12	11
7	18	30
8	24	47
9	32	122
10	40	225
11	50	812
12	60	1628
13	72	7005

3. Closure operations on rings and monoids.

We would like to determine the seminormalization ^+A and weak normalization $^*A$ of the monomial subring A in $\mathbb{Z}[t_1, \dots, t_n]$. In this section we begin the discussion of ^+A and $^*A$, and review the corresponding properties of the integral closure $\widetilde{A}$ of A in $\mathbb{Z}[t_1, \dots, t_n]$.

Basic references for ^+A and $^*A$ are [16] and [18] respectively. These concepts are also reviewed and adapted to the graded case in some of our earlier papers, for example [12] and [11]. The ring ^+A is obtained from A as a filtered union of extensions $A_{i-1} \subseteq A_i = A_{i-1}[x_i]$ where $x_i^2, x_i^3 \in A_{i-1}$. From this it is clear that ${}^+A$ is contained in the field of fractions of A (and hence is the "absolute" seminormalization). Similarly ${}^*A$ is obtained by extensions of the form $A_{i-1} \subseteq A_i = A_{i-1}[x_i]$ where $q_i x_i, x_i^{q_i} \in A_{i-1}$ for some prime number q_i. So long as $\mathbb{Z} \subseteq A$ (as is the case here) it follows that ${}^*A$ is also contained in the quotient field of A, and is the "absolute" weak normalization. (If $A = {}^+A$ then A is called seminormal, and if $A = {}^*A$ then A is called weakly normal.)

Integral closure requires a little more care. The field of fractions K of A is equal to the field of fractions of $\mathbb{Z}[\mathcal{S}]$, since the coefficients can be inverted in K, and also equal to the field of fractions of $\mathbb{Z}[\mathbb{Z}\mathcal{S}]$. The abelian group $\mathbb{Z}\mathcal{S}$ is free of rank $m \leq n$, and (possibly after a change of variables) has a basis of the form $f_i = b_i\mathbf{e}_i$ ($1 \leq i \leq m$, $b_i \in \mathbb{N}$), where $\mathbf{e}_i = (0, 0, \ldots, 1, 0, \ldots 0)$ (1 in the i^{th} place, 0 elsewhere) is a standard basis vector of $\mathbb{R}^n$. If we set $u_i = t_i^{b_i}$ then $A \subseteq \mathbb{Z}[u_1, \ldots, u_m]$ and the field of fractions of A is equal to the field of fractions of $\mathbb{Z}[u_1, \ldots, u_m]$. The integral closure of A in $\mathbb{Z}[t_1, \ldots, t_n]$ is equal to the integral closure of A in $\mathbb{Z}[u_1, \ldots, u_m]$. If we wished we could replace $\mathbb{Z}[t_1, \ldots, t_n]$ by $\mathbb{Z}[u_1, \ldots, u_m]$, but will not do so unless it changes the statements of our theorems, or their proofs. We will work most of the time regarding A as a subring of the original polynomial ring $\mathbb{Z}[t_1, \ldots, t_n]$ and will denote the integral closure of A in $\mathbb{Z}[t_1, \ldots, t_n]$ by $\widetilde{A}$. Depending on the choice of $\mathbb{Z}[t_1, \ldots, t_n]$, $\widetilde{A}$ might or might not be the ordinary integral closure $\overline{A}$ of A (i.e. the integral closure of A in its field of fractions).

By [11, 2.5], respectively [11, 2.7] or [12, 4.1], we have that ${}^+A$ (respectively ${}^*A$) is graded and can be obtained by successive adjunction of terms. More precisely we have the following:

Lemma 3.1. *Grade A by $\mathbb{N}^n$. Then*

(a) *The integral closure $\widetilde{A}$ of A in $\mathbb{Z}[t_1, \ldots, t_n]$ is $\mathbb{N}^n$-graded (in a manner compatible with the grading on A). More generally if $R \subseteq S$ is any inclusion of $\mathbb{N}^n$-graded rings, then the integral closure of R in S is $\mathbb{N}^n$-graded.*

(b) *${}^+A$ is also graded by $\mathbb{N}^n$ (in a manner compatible with the grading on A). Given any homogeneous element s of ${}^+A$ there is a finite sequence of extensions $A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_r$ such that $s \in A_r$ and $A_{i+1} = A_i[x_i]$ where x_i is homogeneous (i.e. a term) with $x_i^2, x_i^3 \in A_i$ $(0 \leq i \leq r-1)$.*

(c) *${}^*A$ is also graded by $\mathbb{N}^n$ (in a manner compatible with the grading on A). Given any homogeneous element s of ${}^*A$ there is a finite sequence of extensions $A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_r$ such that $s \in A_r$ and $A_{i+1} = A_i[x_i]$ where x_i is homogeneous (i.e. a term) with either $x_i^2, x_i^3 \in A_i$ or $p_i x_i, x_i^{p_i} \in A$ (p_i a prime) $(0 \leq i \leq r-1)$.*

Proof. Part (a) can be seen as follows: for any $\mathbb{N}^n$-graded ring R define a homomorphism $\phi : R \to R[t_1, \ldots, t_n]$ by $\phi(x) = xt_1^{m_1}t_2^{m_2}\ldots t_n^{m_n}$, if x is of degree $(m_1, \ldots, m_n)$. Extend by linearity to all of R. Then we have a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\phi_R"] & S \arrow[d, "\phi_S"] \\ R[t_1, \ldots, t_n] \arrow[r] & S[t_1, \ldots, t_n] \end{tikzcd}$$

If $f \in S$ is integral over R then $\phi_S(f)$ is integral over $R[t_1, \ldots, t_n]$. By [3, Proposition 12, page 18] the coefficients of $\phi_S(f)$ are integral over R. But these are the homogeneous components of f. From this it follows that the integral closure of R in S is $\mathbb{N}^n$-graded. (This argument is from [3, page 30], which does only the $n = 1$ case.)

Parts (b) and (c) follow from the proofs of [11, 2.5, 2.7] respectively. □

Theorem 3.2. *Let A be a monomial subring of $\mathbb{Z}[t_1, \ldots, t_n]$ and let $T = a_{\mathbf{M}}\mathbf{t}^{\mathbf{M}} \in \mathbb{Z}[t_1, \ldots, t_n]$ be a term. Then*
(a) *$T \in \widetilde{A}$ if and only if $T^r \in A$ for some $r > 0$.*
(b) *$T \in {}^{+}A$ if and only if $T^r \in A$ for all $r >> 0$.*
(c) *$T \in {}^{*}A$ if and only if there is an integer-valued polynomial $f \in \mathbb{Q}[x]$ such that $f(0) = 1$ and $f(r)T^r \in A$ for $r >> 0$.*

Proof. (a) This is easy if $a_{\mathbf{M}} = 1$ but turned out to be more difficult than expected for general $a_{\mathbf{M}}$! The "if" direction is obvious so we prove just the "only if" part. Let $A = \mathbb{Z}[\{a_i\mathbf{t}^{\mathbf{m}_i}\}]$, and $T = a\mathbf{t}^{\mathbf{M}}$ ($a, a_i \in \mathbb{Z}$, $\mathbf{M}$, $\mathbf{m}_i \in \mathbb{N}^n, T \in \widetilde{A}$). We wish to show that there is an $r \in \mathbb{N}$ such that $T^r = (a\mathbf{t}^{\mathbf{M}})^r \in A$. Since $a\mathbf{t}^{\mathbf{M}}$ is integral over A, there exist $d \in \mathbb{N}$ and $b_i \in A$ such that $(a\mathbf{t}^{\mathbf{M}})^d - b_1(a\mathbf{t}^{\mathbf{M}})^{d-1} + b_2(a\mathbf{t}^{\mathbf{M}})^{d-2} - \ldots + (-1)^d b_d = 0$. Now A is an $\mathbb{N}^n$-graded ring. Taking the $d\mathbf{M}^{\text{th}}$ component of both sides of the equation,we have $(a\mathbf{t}^{\mathbf{M}})^d - (\lambda_1\mathbf{t}^{\mathbf{M}})(a\mathbf{t}^{\mathbf{M}})^{d-1} + (\lambda_2\mathbf{t}^{2\mathbf{M}})(a\mathbf{t}^{\mathbf{M}})^{d-2} - \ldots + (-1)^d(\lambda_d\mathbf{t}^{d\mathbf{M}}) = 0$, with $\lambda_i \in \mathbb{Z}, \lambda_i\mathbf{t}^{i\mathbf{M}} \in A$ for $j = 1, \ldots, d$. Computing the coefficient of $\mathbf{t}^{d\mathbf{M}}$, we have $a^d - \lambda_1 a^{d-1} + \lambda_2 a^{d-2} - \ldots + (-1)^d\lambda_d = 0$ (in $\mathbb{Z}$). Let $f(x) = x^d - \lambda_1 x^{d-1} + \lambda_2 x^{d-2} - \ldots + (-1)^d\lambda_d$, x an indeterminate. Since $f(a) = 0$, we have $f(x) = (x-a)(x^{d-1} - \mu_1 x^{d-2} + \mu_2 x^{d-3} - \ldots + (-1)^{d-1}\mu_{d-1})$. Expanding and equating coefficients we find that $\lambda_1 = a + \mu_1, \lambda_i = a\mu_{i-1} + \mu_i$ for $i = 2, \ldots, d-1$, and $\lambda_d = a\mu_{d-1}$.

Let r=LCM($\{1, 2, \ldots, d\}$). We claim that GCD($\{\lambda_i^{r/i}\}$) $\mid a^r$. For any prime number p and $x \in \mathbb{Z}$, let $\nu_p(x) = \max(\{i \mid p^i \text{ divides } x\})$. To prove our claim it suffices to show that $\min(\{(r/i)\nu_p(\lambda_i)\}) \le r\nu_p(a)$ for all primes p. There are d cases to consider:

Case $1 : \nu_p(\mu_1) \neq \nu_p(a)$
Case $j :$ $(j = 2, \ldots, d-1)$ $\nu_p(\mu_k) = k\nu_p(a), k = 1, \ldots, j-1$, and $\nu_p(\mu_j) \neq j\nu_p(a)$
Case $d : \nu_p(\mu_k) = k\nu_p(a), k = 1, \ldots, d-1$

In case $1, \min(\{(r/i)\nu_p(\lambda_i)\}) \le r\nu_p(\lambda_1) = r\nu_p(a + \mu_1) = r\min(\nu_p(a), \nu_p(\mu_1)) \le r\nu_p(a)$. In case $j, \nu_p(a\mu_{j-1}) = \nu_p(a) + \nu_p(\mu_{j-1}) = \nu_p(a) + (j-1)\nu_p(a) = j\nu_p(a)$. Therefore $\min(\{(r/i)\nu_p(\lambda_i)\}) \le (r/j)\nu_p(\lambda_j) = (r/j)\nu_p(a\mu_{j-1} + \mu_j) = (r/j)\min(j\nu_p(a), \nu_p(\mu_j)) \le (r/j)(j\nu_p(a)) = r\nu_p(a)$. Finally in case d, $\min(\{(r/i)\nu_p(\lambda_i)\}) \le (r/d)\nu_p(\lambda_d) = (r/d)\nu_p(a\mu_{d-1}) = (r/d)(\nu_p(a) + \nu_p(\mu_{d-1})) = (r/d)(\nu_p(a) + (d-1)\nu_p(a)) = (r/d)(d\nu_p(a)) = r\nu_p(a)$. This completes the proof of the claim.

Since GCD($\{\lambda_i^{r/i}\}$) $\mid a^r$, we can find $\alpha_i \in \mathbb{Z}$ such that $\sum_{i=1}^{d} \alpha_i\lambda_i^{r/i} = a^r$. But since $\lambda_i\mathbf{t}^{i\mathbf{M}} \in A$, we have $T^r = (a\mathbf{t}^{\mathbf{M}})^r = \sum_{i=1}^{d} \alpha_i(\lambda_i\mathbf{t}^{i\mathbf{M}})^{r/i} \in A$.

(b) First we prove the following preliminary lemma:

Lemma 3.3. *Let A_1 be a monomial subring of $\mathbb{Z}[t_1, \ldots, t_n]$ and x a term of $\mathbb{Z}[t_1, \ldots, t_n]$. If $A_2 = A_1[x]$ with $x^2, x^3 \in A_1$ and T is a term in A_2 then $T^2, T^3 \in A_1$.*

Proof. Let $T = a + bx$ with $a, b \in A_1$. The result is trivially true if $a = 0$ so assume that $a \neq 0$. Clearly, $(bx)^2, (bx)^3 \in A_1$. We may take $a = \lambda y$ with $\lambda \in \mathbb{Z}$ and y a monomial in $\mathbb{Z}[t_1, \ldots, t_n]$. In order for T to be a term, we must have $bx = \mu y$

with $\mu \in \mathbb{Z}$. Let d=GCD(λ, μ), so that $T = d(\lambda_1 + \mu_1)y$ with GCD$(\lambda_1, \mu_1) = 1$. We are given that $d\lambda_1 y, (d\mu_1 y)^2, (d\mu_1 y)^3 \in A_1$. Since λ_1 and μ_1 are relatively prime, so are λ_1^2, μ_1^2 and λ_1^3, μ_1^3, and therefore there exist $p_2, q_2, p_3, q_3 \in \mathbb{Z}$ such that $p_2\lambda_1^2 + q_2\mu_1^2 = 1$ and $p_3\lambda_1^3 + q_3\mu_1^3 = 1$. But then $(dy)^2 = p_2(d\lambda_1 y)^2 + q_2(d\mu_1 y)^2 \in A_1$ and $(dy)^3 = p_3(d\lambda_1 y)^3 + q_3(d\mu_1 y)^3 \in A_1$ and hence $T^2, T^3 \in A_1$. □

Now we complete the proof of Theorem 3.2(b). Let $A = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_d$ be a sequence of elementary extensions by terms, as in Lemma 3.1(b) and let T be a term of A_d. Then repeatedly applying Lemma 3.3 we find that $T^{2^d}, T^{3^d} \in A_0 = A$. Since 2^d and 3^d are relatively prime, all sufficiently large r (in fact, all $r \geq (2^d - 1)(3^d - 1)$) can be written as a (non-negative integer) linear combination of $2^d$and 3^d and hence $T^r \in A$ for all $r >> 0$. This implies the "only if" part of (b). The "if" part is obvious, so Theorem 3.1(b) follows.

(c) By [14, 6.10] the term $T \in \mathbb{Z}[t_1, \ldots, t_n]$ lies in $^*A$ if and only if T has a system of subintegrality, i.e. if and only if there exists an integer $p \geq 0$, and elements $x_1, \ldots, x_p \in \mathbb{Z}[t_1, \ldots, t_n]$ such that $T^m + \Sigma_{i=1}^p x_i \binom{m}{i} T^{m-i} \in A$ for $m >> 0$. If $T = c_{\mathbf{M}}\mathbf{t}^{\mathbf{M}}$ where $\mathbf{t}^{\mathbf{M}}$ is a monomial of $\mathbb{Z}[t_1, \ldots, t_n]$, then by looking in the $\mathbf{t}^{m\mathbf{M}}$ component we can assume that $x_i = \lambda_i \mathbf{t}^{i\mathbf{M}}$ $(\lambda_i \in \mathbb{Z})$. The condition that T be an element of $^*A$ is then equivalent to the existence of integers λ_i $(1 \leq i \leq p)$ such that $(\Sigma_{i=0}^p \lambda_i \binom{m}{i}) T^m \in A$ for all sufficiently large m $(\lambda_0 = 1)$. Then take $f(m) = \Sigma_{i=0}^p \lambda_i \binom{m}{i}$. All integer-valued polynomials $f(m)$ are of this form, so the proof of Theorem 3.2(c) is complete. □

Recall the observation in Lemma 1.4 that for any monomial subring A of $\mathbb{Z}[t_1, \ldots, t_n]$ we have $A = \cap A^{[q]}$, the intersection taken over all primes q. This is used repeatedly in the following Corollary of Theorem 3.2.

Corollary 3.4. *Let A be a monomial subring of $\mathbb{Z}[t_1, \ldots, t_n]$. Then*

(a) *$\overline{A} = \cap \overline{A^{[q]}}$ and A is normal if and only if all of the $A^{[q]}$ are. We also have $\widetilde{A} = \cap \widetilde{A^{[q]}}$ and A is integrally closed in $\mathbb{Z}[t_1, \ldots, t_n]$ if and only if all of the $A^{[q]}$ are. Furthermore $\overline{A^{[q]}} = \overline{A}^{[q]}$ and $\widetilde{A^{[q]}} = \widetilde{A}^{[q]}$.*

(b) *$^+A = \cap(^+(A^{[q]}))$ and A is seminormal if and only if all the $A^{[q]}$ are. Furthermore $(^+A)^{[q]} = {}^+(A^{[q]})$ for all primes q.*

(c) *$^*A = \cap(^*(A^{[q]}))$ and A is weakly normal if and only if all of the $A^{[q]}$ are. Furthermore $(^*A)^{[q]} = {}^*(A^{[q]})$ for all primes q.*

Proof. The proofs are quite straightforward, using Lemma 1.4 and the appropriate part of Theorem 3.2. We leave the proof of (a) to the reader. We prove only (c) in detail. The proof of (b) is similar by taking f_q all to be 1. Now we prove (c).

We have $A \subseteq A^{[q]}$ for all primes q, so $^*A \subseteq {}^*(A^{[q]})$ for all q, and $^*A \subseteq \cap(^*(A^{[q]}))$. Conversely suppose that $T \in \cap(^*(A^{[q]}))$ (T a term). Then for all q, there are integer-valued polynomials $f_q(x) \in \mathbb{Q}[x]$ and integers $N_q \in \mathbb{N}$, such that $f_q(r)T^r \in A^{[q]}$ for $r \geq N_q$. Now, in fact, for all but finitely many q, $T \in A^{[q]}$, so for these q, we can take $f_q = 1$ and $N_q = 1$ without loss of generality. Letting $f(x) = \Pi f_q(x)$ and $N = \max(\{N_q\})$, we clearly have that$f(x)$ is an integer-valued polynomial in $\mathbb{Q}[x]$ with $f(0) = 1$ and for all $q, f(r)T^r \in A^{[q]}$ for $r \geq N$. Therefore $f(r)T^r \in \cap(A^{[q]}) = A$ and hence $T \in {}^*A$ by Theorem 3.2(c) and so $\cap(^*(A^{[q]})) \subseteq {}^*A$.

If $A^{[q]}$ is weakly normal for all q, then $^*A = \cap(^*(A^{[q]})) = \cap(A^{[q]}) = A$, so A is weakly normal. Now suppose that A is weakly normal. For a fixed prime p, given $T \in {}^*(A^{[p]})$, there is an integer-valued polynomial $f(x) \in \mathbb{Q}[x]$ with $f(0) = 1$ such that $f(r)T^r \in A^{[p]}$ for all $r >> 0$. There is a d relatively prime to p such that $dT \in A^{[p']}$ for all $p' \neq p$. We therefore have $f(r)(dT)^r \in \cap(A^{[q]}) = A$, so $dT \in {}^*A = A$. But the fact that d is relatively to p implies that $T \in A^{[p]}$, hence ${}^*(A^{[p]}) = A^{[p]}$ and $A^{[p]}$ is weakly normal.

The final assertion is proved as follows: we have $A^{[q]} \subseteq ({}^*A)^{[q]}$, and by the above $({}^*A)^{[q]}$ is weakly normal. Therefore ${}^*(A^{[q]}) \subseteq ({}^*A)^{[q]}$. On the other hand we have

(α) $T \in ({}^*A)^{[q]}$ if and only if there exists $d \in \mathbb{N}$ relatively prime to q so that $dT \in {}^*A$ if and only if there is an integer-valued polynomial f with $f(0) = 1$ and d relatively prime to q such that $f(r)(dT)^r \in A$ for $r >> 0$.

Also we have

(β) $T \in {}^*(A^{[q]})$ if and only if there is an integer-valued polynomial g with $g(0) = 1$ so that $g(r)T^r \in A^{[q]}$ for $r >> 0$ if and only if for each $r >> 0$ there exists $d_r \in \mathbb{N}$ relatively prime to q such that $d_r g(r)T^r \in A$.

Setting $d_r = d^r$ and $g = f$ we see that condition (α) implies condition (β), so $({}^*A)^{[q]} \subseteq {}^*(A^{[q]})$. $\square$

If all the coefficients $a_{\mathbf{M}}$ are 1, then A is the monoid ring $\mathbb{Z}[\mathcal{S}]$ and from Theorem 3.2 we recover known properties of monoid rings. First recall the following definitions:

Definition 3.5. If $\mathcal{T}$ is a submonoid of $\mathbb{N}^n$ then we will denote the normalization of $\mathcal{T}$ (i.e. $\{\mathbf{M} \in \mathbb{Z}\mathcal{T} \mid m\mathbf{M} \in \mathcal{T}$ for some $m \in \mathbb{Z}_+\}$) by $\overline{\mathcal{T}}$. Let $\widetilde{\mathcal{T}} = \{\mathbf{M} \in \mathbb{Z}^n \mid m\mathbf{M} \in \mathcal{T}$ for some $m \in \mathbb{Z}_+\}$. If $\mathcal{T}$ is any subset of $\mathbb{N}^n$ we will denote by $\mathcal{C}(\mathcal{T})$ the cone of $\mathcal{T}$, i.e. the set of all linear combinations $\Sigma\lambda_i\mathbf{M}_i$ with $\lambda_i \geq 0$ real, $\mathbf{M}_i \in \mathcal{T}$.

Definition 3.6. Let $\mathcal{T}$ be a submonoid of $\mathbb{N}^n$. Then the seminormalization of $\mathcal{T}$ is ${}^+\mathcal{T} := \{\mathbf{M} \in \mathbb{Z}\mathcal{T} \mid m\mathbf{M} \in \mathcal{T}$ for all $m \in \mathbb{N}, m >> 0\}$ $(= \{\mathbf{M} \in \mathbb{Z}^n \mid m\mathbf{M} \in \mathcal{T}$ for all $m \in \mathbb{N}, m >> 0\})$. (The latter equality holds because if $m\mathbf{M} \in \mathcal{T}, (m+1)\mathbf{M} \in \mathcal{T}$, then $\mathbf{M} \in \mathbb{Z}\mathcal{T}$.)

Definition 3.5 is well known, for example it occurs in [7, page 151] (there called "integral closure"). Definition 3.6 is perhaps less well known, but we found it in several places in the literature, for example [10, Definition 2.1]. Both definitions also occur in [17, Section 4]. Note that the "weak normalization" of a monoid is the same as the seminormalization as we will see in Theorem 3.8(b) below.

Remark 3.7. $\widetilde{\mathcal{T}} = \mathcal{C}(\mathcal{T}) \cap \mathbb{Z}^n$ and $\overline{\mathcal{T}} = \mathcal{C}(\mathcal{T}) \cap \mathbb{Z}\mathcal{T}$.

Theorem 3.8. (a) *The integral closure of $\mathbb{Z}[\mathcal{T}]$ in its total ring of fractions is $\mathbb{Z}[\overline{\mathcal{T}}]$ and the integral closure of $\mathbb{Z}[\mathcal{T}]$ in $\mathbb{Z}[t_1, \ldots, t_n]$ is $\mathbb{Z}[\widetilde{\mathcal{T}}]$.*
(b) *The seminormalization and weak normalization of $\mathbb{Z}[\mathcal{T}]$ (in its total ring of fractions or in $\mathbb{Z}[\mathbb{Z}\mathcal{T}]$ or in $\mathbb{Z}[t_1, \ldots, t_n]$) are all equal to $\mathbb{Z}[{}^+\mathcal{T}]$.*

Proof. (a) This follows from Lemma 3.1(a) and Theorem 3.2(a).
(b) This follows from Lemma 3.1(b,c) and Theorem 3.2(b,c). In Theorem 3.2(c) we can take $f(r) = 1$, and the condition for a term T to be in ${}^*A$ is the same as to be in ${}^+A$. $\square$

Theorem 3.8(a) can be found in [7, Corollary 12.11] and Theorem 3.8(b) follows from [1, Corollary 6.2].

Definition 3.9. Let $A \subseteq \mathbb{Z}[t_1, \ldots, t_n]$ be a monomial subring with support $\mathcal{S}$. If $\mathbf{M} \in \mathbb{N}^n$ we define the ray of $\mathbf{M}$ to be $\mathcal{R}(\mathbf{M}) = \mathcal{C}(\{\mathbf{M}\}) \cap \mathbb{Z}\mathcal{S}$.

Now $\mathcal{R}(\mathbf{M})$ is a submonoid of $\mathbb{N}^n$. Obviously $\mathcal{R}(\mathbf{M}) = \mathcal{R}(x\mathbf{M})$ for any positive integer x, and $\mathcal{R}(\mathbf{M})$ is the set of all non-negative integer multiples of some element $\mathbf{U}$ of $\mathbb{N}^n$, so $\mathbb{Z}[\mathcal{R}(\mathbf{M})] \cong \mathbb{Z}[u]$, where $u = \mathbf{t}^{\mathbf{U}}$ ($\mathbf{U}$ will be $\mathbf{0}$ and u will be 1 if $\mathbf{M} \notin \mathcal{C}(\mathcal{S})$, in which case $\mathbb{Z}[\mathcal{R}(\mathbf{M})] \cong \mathbb{Z}$).

Definition 3.10. Let A be a monomial subring of $\mathbb{Z}[t_1, \ldots, t_n]$. With notation as above, define $A(\mathbf{M}) \subseteq \mathbb{Z}[\mathcal{R}(\mathbf{M})]$ to be the subring generated by $\{a_{i\mathbf{U}}\mathbf{t}^{i\mathbf{U}}\}_{i\in\mathbb{N}}$. (Note that $A(\mathbf{M})$ is a diagonal subring of A in the language of [5] or a straight-line subalgebra in the language of [15].)

Examples of $\mathcal{R}(\mathbf{M})$ and $A(\mathbf{M})$ are given in section 5.

Theorem 3.2 says that $\widetilde{A}, {}^+A$ and ${}^*A$ can be computed raywise:

Theorem 3.11. *Let A be a monomial subring of $\mathbb{Z}[t_1, \ldots, t_n]$ and let $\mathbf{M}$ be a monomial. Then $(\widetilde{A})(\mathbf{M}) = (A(\mathbf{M}))\tilde{}$, $({}^+A)(\mathbf{M}) = {}^+(A(\mathbf{M}))$ and $({}^*A)(\mathbf{M}) = {}^*(A(\mathbf{M}))$.* $\square$

In some sense this reduces the calculation of ${}^+A$ and ${}^*A$ to the case $n = 1$, which was studied in [12]. However the rays are difficult to deal with, and are not independent of each other, in the sense that adding elements to $A(\mathbf{M})$ may change A elsewhere.

Remark 3.7 gives the monoid $\widetilde{\mathcal{T}}$ a simple geometric description. In the next section we will give a similar geometric description of ${}^+\mathcal{T}$.

4. Geometric interpretation of seminormalization and weak normalization.

In this section we first describe (in Theorem 4.3) a geometric interpretation of the seminormalization of a finitely generated submonoid of $\mathbb{N}^n$. We use this to give a more geometric interpretation of seminormalization and weak normalization of our monomial subring A in the case where the cone of 1's is finitely generated and equal to the cone of the support (Theorems 4.10 and 4.11). We conclude the section with a discussion of the seminormalization of submonoids of $\mathbb{N}^n$ that are not finitely generated. Examples are given in the next section.

First we review some properties of monoids. A convenient reference is [17]. First we recall the following definition:

Definition 4.1. [17, page 225] Let $\mathcal{S}$ be a monoid. Then $\mathrm{Int}(\mathcal{S}) = \{x \in \mathcal{S} \mid$ for all $y \in \mathcal{S}$ we can find $n \in \mathbb{N}$ with $nx = y + z,\ z \in \mathcal{S}\}$.

Note that Swan's monoids are written multiplicatively, whereas ours are additive.) In the finitely generated case $\mathrm{Int}(\mathcal{S})$ is just $\mathcal{S} \cap \mathring{\tau}$ where $\tau = \mathcal{C}(\mathcal{S})$ and $\mathring{\tau}$ denotes the relative interior of the cone τ.

Parts (a) and (b) of the following Lemma are standard facts about monoids, and (c) is a finitely generated version of a result of Gubeladze ([8]).

Lemma 4.2. *(a) Let $\mathcal{T}$ be a finitely generated submonoid of $\mathbb{N}^n$ and $\sigma = \mathcal{C}(\mathcal{T})$. Suppose that τ is a d-dimensional face of σ. Then $\mathcal{C}(\mathcal{T} \cap \tau) = \tau$ (which implies that $\mathbb{Z}(\mathcal{T} \cap \tau)$ is of rank d).*

(b) Let $\mathcal{T}$ be a finitely generated submonoid of $\mathbb{N}^n$ and $\sigma = \mathcal{C}(\mathcal{T})$. Suppose that $\mathbb{Z}\mathcal{T}$ is of rank n. Then the submonoid ${}^+\mathcal{T}$ is also finitely generated.

(c) If $\mathcal{S}$ is a cancellative, torsion free, seminormal, finitely generated monoid, then $\mathrm{Int}(\mathcal{S}) \cup \{\mathbf{0}\}$ is normal.

Proof. (a) This follows from the observation that τ is spanned by elements of $\mathcal{T}$.

(b) This is [17, Corollary 4.3]. It also follows from Corollary 4.8 below (the proof of which is independent of the present discussion).

(c) This is [17, Lemma 6.5]. □

The hypotheses in Lemma 4.2(c) that $\mathcal{S}$ be cancellative, torsion free and finitely generated are equivalent to $\mathcal{S}$ being a finitely generated submonoid of $\mathbb{Z}^n$ for some $n \in \mathbb{N}$.

We now have:

Theorem 4.3. *Let $\mathcal{T}$ be a finitely generated submonoid of $\mathbb{N}^n$. Then ${}^+\mathcal{T} = \cup[(\mathbb{Z}(\mathcal{T} \cap \tau)) \cap \mathring{\tau}]$ where τ ranges over all faces of $\mathcal{C}(\mathcal{T})$.*

Proof. Obviously we have ${}^+\mathcal{T} = \cup(({}^+\mathcal{T}) \cap \mathring{\tau})$ as τ ranges over all faces of $\mathcal{C}(\mathcal{T})$. But $({}^+\mathcal{T}) \cap \tau = {}^+(\mathcal{T} \cap \tau)$ by Definition 3.6, and $\mathrm{Int}^+(\mathcal{T} \cap \tau) = ({}^+\mathcal{T}) \cap \mathring{\tau}$ by Lemma 4.2(a) (and the remarks on Int after Definition 4.1). Hence $(({}^+\mathcal{T}) \cap \mathring{\tau}) \cup \{\mathbf{0}\}$ is normal by Lemma 4.2(c). By Remark 3.7 we have $(({}^+\mathcal{T}) \cap \mathring{\tau}) \cup \{\mathbf{0}\} = (\mathbb{Z}({}^+\mathcal{T} \cap \tau) \cap \mathring{\tau}) \cup \{\mathbf{0}\}$. The latter equals $(\mathbb{Z}(\mathcal{T} \cap \tau) \cap \mathring{\tau}) \cup \{\mathbf{0}\}$ so $(({}^+\mathcal{T}) \cap \mathring{\tau}) = (\mathbb{Z}(\mathcal{T} \cap \tau)) \cap \mathring{\tau}$, completing the proof. □

We now have:

Remark 4.4. All finitely generated rank n seminormal submonoids of $\mathbb{N}^n$ can be constructed in the following manner: start with a finitely generated rational cone $\mathcal{C}$ of dimension n. If τ is a d-dimensional face of $\mathcal{C}$ then the linear subspace of $\mathbb{R}^n$ spanned by τ intersects the lattice $\mathbb{Z}^n$ in a free abelian group $F(\tau)$ of rank d. If $\tau_1 \subseteq \tau_2$ there is an obvious inclusion $F(\tau_1) \subseteq F(\tau_2)$. Define our monoid $\mathcal{S}$ as follows: for all faces τ choose a rank d subgroup $G(\tau) \subseteq F(\tau)$, subject to the compatibility conditions that if $\tau_1 \subseteq \tau_2$ then $G(\tau_1) \subseteq G(\tau_2) \cap F(\tau_1)$. Then take $\mathcal{S} \cap \mathring{\tau}$ to be $G(\tau) \cap \mathring{\tau}$. The proposed monoid $\mathcal{S}$ is closed under addition (hence really is a monoid) since the monoid elements on each subface are fewer than "expected", and $\mathcal{S}$ is seminormal by Theorem 4.3.

As the first step in a global approach we introduce:

Definition 4.5. Let A be a monomial subring of $\mathbb{Z}[t_1, \ldots, t_n]$. Then we define the "monoid of ones" $\mathcal{O}$ to be $\{\mathbf{M} \in \mathcal{S} \mid \mathbf{t}^{\mathbf{M}} \in A\}$. For each prime q we also define the monoid $\mathcal{O}_q$ to be $\{\mathbf{M} \in \mathcal{S} \mid x\mathbf{t}^{\mathbf{M}} \in A$ for some $x \in \mathbb{Z}_+$ such that $q \nmid x\}$.

Note that $\mathcal{O}_q = \mathcal{O}_{A^{[q]}}$.

Example 4.6. Let $\mathcal{S}$ be the first quadrant with the t_1-axis omitted, i.e. $\mathcal{S} = \{(x, y) \in \mathbb{N}^2 \mid 0 \leq x, 0 < y\}$. Let $\mathcal{O} = \{(x, y) \mid 0 \leq x, y \geq x\}$ and $\mathcal{O}_q = \{(x, y) \mid 0 \leq x, y \geq x/(q-1)\}$ for each prime number q. Define the ring A by taking $a_{(x,y)}$ to be the product of all q such that $(x, y) \notin \mathcal{O}_q$ (so that $\mathcal{O}_q$ as defined above really ends up as the $\mathcal{O}_q$ associated to A). Then all primes q occur as factors of the coefficients of A and none of the $A^{[q]}$ is a finitely generated $\mathbb{Z}$-algebra. This example shows that some sort of finiteness assumption is desirable.

We have $\mathcal{O} \subseteq \mathcal{O}_q$ for all q and $\mathcal{O} = \cap\mathcal{O}_q$. Obviously $\mathcal{C}(\mathcal{O}) \subseteq \mathcal{C}(\mathcal{O}_q) \subseteq \mathcal{C}(\mathcal{S})$. In this section we will assume for simplicity that $\mathcal{C}(\mathcal{O}) = \mathcal{C}(\mathcal{S})$, and furthermore that $\mathcal{C}(\mathcal{S})$ is a finitely generated rational cone. These assumptions simplify things quite a lot. For example A will then be a finitely generated $\mathbb{Z}$-algebra:

Theorem 4.7. *Let $A \subseteq \mathbb{Z}[t_1, \ldots, t_{n'}]$ be a monomial subring with support $\mathcal{S}$. Suppose that $\mathcal{C}(\mathcal{S}) = \mathcal{C}(\mathcal{O})$ is a finitely generated rational cone of dimension n $(\leq n')$. Then A is a finitely generated $\mathbb{Z}$-algebra.*

Proof. Suppose that $\mathcal{C}(\mathcal{S})$ is spanned by $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$. For each n-element linearly independent subset $\mathcal{P}$ of $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ let $A_{\mathcal{P}}$ be the subring of A generated by all the terms $x_{\mathbf{M}}t^{\mathbf{M}}$ such that $\mathbf{M} \in \mathcal{P}$. By Carathéodory's Theorem ([19, page 46]) every element of $\mathcal{C}(\mathcal{S})$ lies in the cone spanned by n linearly independent vectors in $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$. Thus the union of the generators of the $A_{\mathcal{P}}$ generates A, so it suffices to consider the case where $\mathcal{C}(\mathcal{S})$ is spanned by n linearly independent vectors. Furthermore by a change of co-ordinates we can assume that $\mathcal{C}(\mathcal{O}) = \mathcal{C}(\mathcal{S}) = \mathbb{R}^n_{\geq 0}$ and $\mathbb{Z}\mathcal{S} = \mathbb{Z}^n$. Now the standard basis vectors $\mathbf{e}_i$ are in $\mathcal{C}(\mathcal{O})$. Hence there exists $d_i \in \mathbb{N}$ so that $d_i\mathbf{e}_i \in \mathcal{O}$, and $t_i^{d_i} \in A$ $(1 \leq i \leq n)$. Thus $B = \mathbb{Z}[t_1^{d_1}, \ldots, t_n^{d_n}] \subseteq A$. Clearly $\mathbb{Z}[t_1, \ldots, t_n]$ is a finitely generated B module, and A is a B-submodule of $\mathbb{Z}[t_1, \ldots, t_n]$. Since B is Noetherian, A is a finitely generated B-module. The B-module generators of A, together with $\{t^{d_i}\}_{1 \leq i \leq n}$, generate A as a $\mathbb{Z}$-algebra, completing the proof of Theorem 4.7. □

In the special case where all the coefficients are 1 we recover the following well-known result (which for example is immediate from [17, Theorem 4.2]):

Corollary 4.8. *Let $\mathcal{S}$ be a submonoid of $\mathbb{N}^n$ such that $\mathcal{C}(\mathcal{S})$ is a finitely generated rational cone. Then $\mathcal{S}$ is a finitely generated monoid.*

Proof. By Theorem 4.7, $\mathbb{Z}[\mathcal{S}]$ is a finitely generated $\mathbb{Z}$-algebra. Hence $\mathcal{S}$ is a finitely generated monoid. □

Theorem 4.9. *Let A be a monomial subring of $\mathbb{Z}[t_1, \ldots, t_n]$ such that $\mathcal{C}(\mathcal{O}) = \mathcal{C}(\mathcal{S})$. Then $\overline{A} = \mathbb{Z}[\overline{\mathcal{S}}]$ and $\widetilde{A} = \mathbb{Z}[\widetilde{\mathcal{S}}]$.*

Proof. First we check that for each $\mathbf{M} \in \mathcal{S}\backslash\{\mathbf{0}\}$ the ray $\mathcal{R}(\mathbf{M})$ intersects $\mathcal{O}$ non-trivially. Since $\mathcal{C}(\mathcal{S}) = \mathcal{C}(\mathcal{O})$ there are elements $\mathbf{M}_1, \ldots, \mathbf{M}_r \in \mathcal{O}$ and non-negative rational numbers s_i such that $\mathbf{M} = \Sigma_{i=1}^r s_i\mathbf{M}_i$. Clearing denominators yields an element of $\mathcal{R}(\mathbf{M}) \cap \mathcal{O}$. Now $\mathbb{Z}[\mathcal{R}(\mathbf{M})] \cong \mathbb{Z}[u]$ and $u^n \in A(\mathbf{M})$ for some $n \in \mathbb{N}$. Therefore $\mathbb{Z}[\mathcal{R}[\mathbf{M}]]$ is integral over $A(\mathbf{M})$, i.e. $A(\mathbf{M})\tilde{} = \mathbb{Z}[\mathcal{R}[\mathbf{M}]]$, and by Theorem 3.11, $\widetilde{A} = \mathbb{Z}[\widetilde{\mathcal{S}}]$. We can show that $\overline{A} = \mathbb{Z}[\overline{\mathcal{S}}]$ by changing variables so that $\overline{A} = \widetilde{A}$, or equivalently by working in the sublattice $\mathbb{Z}\mathcal{S}$. □

Theorem 4.10. *Let A be a monomial subring of $\mathbb{Z}[t_1, \ldots, t_n]$ such that $\mathcal{C}(\mathcal{O}) = \mathcal{C}(\mathcal{S})$. Then ${}^+A$ is obtained as follows: at each element $\mathbf{M}$ of ${}^+\mathcal{O}$ take the coefficient ${}^+a_M$ of t^M to be 1, and if $\mathbf{M} \in {}^+\mathcal{S}$, $\mathbf{M} \notin {}^+\mathcal{O}$ take the coefficient ${}^+a_{\mathbf{M}}$ of $\mathbf{t}^{\mathbf{M}}$ to be the product of all primes q such that $\mathbf{M} \notin {}^+\mathcal{O}_q$.*

Proof. By ${}^+\mathcal{O}$, respectively ${}^+\mathcal{O}_q$, we mean the seminormalization of $\mathcal{O}$, respectively $\mathcal{O}_q$, as defined in Definition 3.6. As in the proof of Theorem 4.9 each ray $\mathcal{R}(\mathbf{M})$ intersects $\mathcal{O} \subseteq {}^+\mathcal{O} \subseteq {}^+\mathcal{O}_q$. Theorem 4.10 now follows from Theorem 3.11 and the one-dimensional case [12, 3.5, 4.2]. (In particular, the product of primes that we have to form in the construction of ${}^+A$ is finite.)

□

We might also remark that we can construct a seminormal subring of $\mathbb{Z}[t_1, \ldots, t_n]$ by choosing the monoids ${}^+\mathcal{S}$, and ${}^+\mathcal{O}_q$ arbitrarily as in Remark 4.4 subject to the conditions ${}^+\mathcal{O}_q \subseteq {}^+\mathcal{S}$ and that for given $\mathbf{M} \in {}^+\mathcal{S}$ we have $\mathbf{M} \notin {}^+\mathcal{O}_q$ for only finitely many q. Next we globalize the weak normalization construction:

Theorem 4.11. *Let A be a monomial subring of $\mathbb{Z}[t_1, \dots, t_n]$ such that $\mathcal{C}(\mathcal{O}) = \mathcal{C}(\mathcal{S})$. Then the weak normalization $^*A$ of A is obtained from the following construction: first form ^+A as in Theorem 4.10. Let τ be a d-dimensional face of $\mathcal{C}(\mathcal{S})$. For each prime q define $G(\tau)$ to be $\mathbb{Z}(^+\mathcal{S} \cap \tau)$ and $H_q(\tau)$ to be $\mathbb{Z}(^+\mathcal{O}_q \cap \tau)$. As in Lemma 4.3(a) and Remark 4.4, $G(\tau)$ is a rank d subgroup of $\mathbb{Z}^n$ and $H_q(\tau)$ is a rank d subgroup of $G(\tau)$. We can choose a basis of $G(\tau)$ so that (under the co-ordinate isomorphism induced by this basis) $G(\tau) = \mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$ (d times) and $H_q(\tau) = c_1\mathbb{Z} \oplus c_2\mathbb{Z} \oplus \dots \oplus c_d\mathbb{Z}$. Then define $H_q'(\tau) = \widetilde{c}_1\mathbb{Z} \oplus \widetilde{c}_2\mathbb{Z} \oplus \dots \oplus \widetilde{c}_d\mathbb{Z}$ where $c_i = q^{a_i}\widetilde{c}_i$ with $GCD(q, \widetilde{c}_i) = 1$. (This construction is independent of the choice of basis since $H_q'(\tau)$ is the set of elements of $G(\tau)$ which when multiplied by some power of q lie in $H_q(\tau)$.) Now define a subring $^{**}A$ of $\mathbb{Z}[t_1, \dots, t_n]$ by taking coefficient 1 if $\mathbf{M} \in H_q'(\tau) \cap \overset{\circ}{\tau}$ and coefficient q if $\mathbf{M} \in {}^+\mathcal{S} \cap \overset{\circ}{\tau} = G(\tau) \cap \overset{\circ}{\tau}$, but $\mathbf{M} \notin H_q'(\tau) \cap \overset{\circ}{\tau}$ and then multiplying over the various q. We claim that $^{**}A = {}^*A$*

Proof. In the proof of Theorem 4.9 we observed that each ray meets $\mathcal{O}$ (hence $\mathcal{O}_q$). On each ray the coefficient sequence $1\ q\ q\ q\ q\ q\ \dots q\ 1$ of ^+A (which has period cq^r, with $\mathrm{GCD}(q, c) = 1$) has been replaced by $1\ q\ q\ \dots q\ 1$ with period c so in each ray we have the weak normalization as obtained in [12, 4.2]. The result now follows by Theorem 3.11. □

Remark 4.12. It might be helpful now to look back at the one-dimensional case and compare with Theorems 4.10 and 4.11. Let $A = \oplus a_i t^i$, with $a_d = 1$ for some $d \in \mathbb{N}$. Then in [12] we observed that if $i >> 0$ the sequence becomes periodic mod d, and if we extend back to 0 periodically we get a ring $B = \oplus b_i t^i$ (with $b_i = a_i$ for $i >> 0$). Then ^+A is obtained by replacing each coefficient b_i by its "radical", i.e. The interval $[0, d)$ is a "fundamental region" whose translates covers $\mathbb{R}_{\geq 0}$ without overlap. In the case where $n > 1$ (with $\mathcal{C}(\mathcal{O}) = \mathcal{C}(\mathcal{S})$) there is no fundamental region for $\mathcal{C}(\mathcal{S})$ (except in the trivial "simplical cone" case where $\mathcal{C}(\mathcal{S})$ is spanned by n linearly independent vectors). However $\mathbb{Z}\mathcal{S}$ is a free abelian group of rank $m \leq n$ and its fundamental regions do cover $\mathcal{C}(\mathcal{S})$, normally sticking out of the sides. (Alternatively we could take a different fundamental region for $\mathcal{C}(\mathcal{S})$ in each simplicial cone of a covering of $\mathcal{C}(\mathcal{S})$ by simplicial cones, and not have anything stick out.) In the interior of $\mathcal{C}(\mathcal{S})$, A will eventually have periodic coefficients, and the coefficients can be extended back to zero periodically as in the one-dimensional case. Then Theorems 4.10 and 4.11 say that the coefficients of ^+A and $^*A$ in the interior of $\mathcal{C}(\mathcal{S})$ are computed in the same way as with $n = 1$. One repeats the process on the faces of $\mathcal{C}(\mathcal{S})$. The case $n = 1$ is thus seen to be deceptively simple. There the only proper face is $\mathbf{0}$, which does not have to be considered separately, and there is a fundamental region, so the concept of periodicity is more transparent.

Our original proof of Theorem 4.3 used the fact that the conductor of a finitely generated submonoid of $\mathbb{N}^n$ in its normalization is non-zero. The present proof does not use this, but conductors are still of interest when studying submonoids of $\mathbb{N}^n$, so we discuss them briefly.

Definition 4.13. Let $\mathcal{T}_1 \subseteq \mathcal{T}_2$ be monoids. Then $\{\mathbf{f} \in \mathcal{T}_2 \mid \mathbf{f} + \mathcal{T}_2 \subseteq \mathcal{T}_1\}$ is called the conductor of $\mathcal{T}_1$ in $\mathcal{T}_2$.

It is immediate from the definition that the conductor of $\mathcal{T}_1$ in $\mathcal{T}_2$ is actually contained in $\mathcal{T}_1$ and that if the conductor is non-empty then $\mathcal{T}_1$ and $\mathcal{T}_2$ have the same quotient group. Furthermore the conductor (in the usual ring-theoretic sense) of

$\mathbb{Z}[\mathcal{T}_1]$ in $\mathbb{Z}[\mathcal{T}_2]$ is the abelian subgroup of $\mathbb{Z}[\mathcal{T}_1]$ generated by the conductor of $\mathcal{T}_1$ in $\mathcal{T}_2$. The conductor is particularly interesting in the case where $\mathcal{T}_1$ is finitely generated and $\mathcal{T}_2 = \overline{\mathcal{T}_1}$ (in which case the conductor is non-empty because the corresponding conductor of monoid rings is non-empty). An example of the conductor is give in the next Section (Theorem 5.2).

If $\mathcal{S}$ is not finitely generated the conductor of $\mathcal{S}$ in its normalization may be empty. This is illustrated by the following example:

Example 4.14. Let $\mathcal{S}$ be the submonoid of $\mathbb{N}^2$ generated by $\{(2^i,1)\}_{i=0}^{\infty}$. Then the conductor of $\mathcal{S}$ in $\overline{\mathcal{S}}$ is empty.

Proof. We first note that since $(1,0) = (2,1) - (1,1)$ and $(0,1) = 2(1,1) - (2,1)$, $\mathbb{Z}\mathcal{S} = \mathbb{Z}^2$. We also have $\mathcal{C}(\mathcal{S}) = \{(x,y) \mid 0 < y \leq x, x, y \in \mathbb{R}\}$ and hence $\overline{\mathcal{S}} = \mathbb{Z}\mathcal{S} \cap \mathcal{C}(\mathcal{S}) = \{(a,b) \mid 0 < b \leq a, a, b \in \mathbb{N}\}$. To prove the claim, we must show that for all $(a,b) \in \overline{\mathcal{S}}$, there are $(s_1,t_1),(s_2,t_2) \in \overline{\mathcal{S}}\backslash\mathcal{S}$ such that $(a,b)+(s_1,t_1) = (s_2,t_2)$ (for then (a,b) in not in the conductor of $\mathcal{S}$ in $\overline{\mathcal{S}}$). We assert that $(2^n - i, j) \in \overline{\mathcal{S}}\backslash\mathcal{S}$, for all $n \geq 2, 1 \leq i \leq n-1, 1 \leq j \leq n-i$. Assuming the result for now, letting $(s_1,t_1) = (2^{a+2} - a - 1, 1)$ and $(s_2,t_2) = (2^{a+2} - 1, b+1)$ concludes the proof of the claim.

We now prove the assertion in the preceding paragraph. We have $0 < j \leq n-i \leq 2^n - i$, so $(2^n - i, j) \in \overline{\mathcal{S}}$. Since the generators of $\mathcal{S}$ are of the form $(2^i,1)$, we have $(2^n - i, j) \in \mathcal{S}$ if and only if $2^n - i$ can be written as the sum of exactly j powers of 2. To show that $(2^n - i, j) \notin \mathcal{S}$ it will suffice to show that $2^n - i$ cannot be written as the sum of $n - i$ (or fewer) powers of 2. If a positive integer's base 2 representation has k 1's, it cannot be written as a sum of fewer than k powers of 2. But now an easy induction on n shows that the number of 1's in the base 2 representation of $2^n - i$ is greater than $n - i$ (which is in turn greater than or equal to j), and the assertion follows. $\square$

It is also of interest that Theorem 4.3 can be generalized to arbitrary submonoids of $\mathbb{N}^n$. This is done in [8, Proposition 1.6], even with n infinite. See also [9, Lemma 1.4.3]. The referee has suggested the following alternative argument which reduces the case of arbitrary submonoids of $\mathbb{N}^n$ to Lemma 4.2(c) and considerably simplifies an argument we gave in the original version of this paper.

First of all observe that if $\mathcal{S}$ is arbitrary then $\mathrm{Int}(\mathcal{S})$ can be empty, for example if $\mathcal{S}$ is the direct sum of infinitely many copies of $\mathbb{N}$. However if $\mathcal{S}$ is a submonoid of $\mathbb{N}^n$ then an interior point of any maximal rank finitely generated submonid of $\mathcal{S}$ is an interior point of $\mathcal{S}$ so $\mathrm{Int}(\mathcal{S})$ is non-empty. More generally it is straightforward to prove the following:

Lemma 4.15. *Let $\mathcal{S}$ be any submonoid of $\mathbb{N}^n$. Then* $\mathrm{Int}(\mathcal{S})$ *is the filtered union of the interiors of the finitely generated full rank submonoids of $\mathcal{S}$.* $\square$

Now recall the following definition from [17, Section 5]:

Definition 4.16. Let $\mathcal{E}$ be a submonoid of an (additive) monoid $\mathcal{S}$. Then $\mathcal{E}$ is an extremal submonoid of $\mathcal{S}$ if $x, y \in \mathcal{S},\ x + y \in \mathcal{E}$ imply $x, y \in \mathcal{E}$.

Using the observation that $\mathrm{Int}(\mathcal{S})$ is non-empty for any submonoid $\mathcal{S}$ of $\mathbb{N}^n$, the proof of [17, Lemma 5.3] goes through in the not necessarily finitely generated case yielding

Lemma 4.17. *Let $\mathcal{S}$ be an arbitrary submonoid of $\mathbb{N}^n$. Then* $\{\mathrm{Int}(\mathcal{E})\}$*, as $\mathcal{E}$ ranges over the extremal submonoids of $\mathcal{S}$, is a disjoint partition of $\mathcal{S}$.* $\square$

We now have

Theorem 4.18. *Let $\mathcal{S}$ be an arbitrary submonoid of $\mathbb{N}^n$. Then $\mathcal{S}$ is seminormal if and only if* $\mathrm{Int}(\mathcal{E}) \cup \{\mathbf{0}\}$ *is normal for all extremal submonoids $\mathcal{E}$ of $\mathcal{S}$.*

Proof. Suppose that $\mathrm{Int}(\mathcal{E}) \cup \{\mathbf{0}\}$ is normal for all extremal submonoids $\mathcal{E}$ of $\mathcal{S}$. Let $\mathbf{x} \in \mathbb{Z}\mathcal{S}$, with $N\mathbf{x}, (N+1)\mathbf{x} \in \mathcal{S}$ for some $N \in \mathbb{N}$. Then, by Lemma 4.17, $N\mathbf{x} \in \mathrm{Int}(\mathcal{E})$ for a (unique) extremal submonoid $\mathcal{E}$ of $\mathcal{S}$. It follows easily from the definitions of Int and extremal that also $(N+1)\mathbf{x} \in \mathrm{Int}(\mathcal{E})$. Therefore $\mathbf{x} \in \mathbb{Z}(\mathrm{Int}(\mathcal{E}) \cup \{\mathbf{0}\})$, and since $\mathrm{Int}(\mathcal{E}) \cup \{\mathbf{0}\}$ is normal we have $\mathbf{x} \in \mathrm{Int}(\mathcal{E}) \cup \{\mathbf{0}\} \subseteq \mathcal{S}$. Therefore $\mathcal{S}$ is seminormal.

Conversely suppose that $\mathcal{S}$ is seminormal. Then so also is $\mathcal{E}$ for any extremal submonoid $\mathcal{E}$ of $\mathcal{S}$. By Lemma 4.15, $\mathrm{Int}(\mathcal{E}) \cup \{\mathbf{0}\}$ is the filtered union of $\mathrm{Int}(\mathcal{E}') \cup \{\mathbf{0}\}$ over all finitely generated full rank submonoids $\mathcal{E}'$ of $\mathcal{E}$. The seminormalization of any such $\mathcal{E}'$ is finitely generated by Corollary 4.8 so $\mathrm{Int}(\mathcal{E}) \cup \{\mathbf{0}\}$ is the filtered union of $\mathrm{Int}(\mathcal{E}') \cup \{\mathbf{0}\}$ over all finitely generated full rank seminormal submonoids $\mathcal{E}'$ of $\mathcal{E}$. The latter are normal by Lemma 4.2(c), hence so also is $\mathrm{Int}(\mathcal{E}) \cup \{\mathbf{0}\}$. □

Note 4.19. If $\mathcal{T}$ is a finitely generated submonoid of $\mathbb{N}^n$ then the extremal submonoids of $\mathcal{T}$ are the submonoids $\mathcal{T} \cap \sigma$ as σ ranges over all faces of the finitely generated rational cone $\mathcal{C}(\mathcal{T})$ and $\mathrm{Int}(\mathcal{T} \cap \sigma) = \mathcal{T} \cap \mathring{\sigma}$ so Theorem 4.18 generalizes Theorem 4.3. However one has to be careful with cones that are not finitely generated. If a cone $\sigma \subset \mathbb{R}^n$ is finitely generated then a face of σ can be defined as the intersection of σ with a hyperplane $H \subset \mathbb{R}^n$ such that $\sigma \backslash H$ lies only on one side of H. However this is not the case if the cone is not finitely generated. For example let X be the unit square $\{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ with semicircles $\{x \geq 1, (x-1)^2 + (y-1/2)^2 \leq 1/2\}, \{x \leq 0, x^2 + (y-1/2)^2 \leq 1/2\}$ added at the left and right. Let $\sigma \subseteq \mathbb{R}^3$ be the cone on $\{(\mathbf{a}, 1) \mid \mathbf{a} \in X\}$ and $\mathcal{S} = \sigma \cap \mathbb{Z}^3$. Let $\mathbf{v} = (0, 0, 1)$. Then $\mathbf{v}$ generates an extremal submonoid of $\mathcal{S}$ but $\mathbb{R}_{\geq 0}\mathbf{v}$ is not the intersection of σ with any hyperplane of $\mathbb{R}^3$.

5. Examples.

In this section we give examples of rays, seminormalization, weak normalization, and the conductor.

Example 5.1. Let $A = \mathbb{Z}[t^3, tu^2, 2tu]$. Then the table of coefficients of the monomials $t^i u^j$ $(0 \leq i \leq 10, 0 \leq j \leq 10)$ in A is as follows (with $1 = t^0 u^0$ in the lower left corner, the t-axis horizontal and the u-axis vertical):

0	0	0	0	0	1	4	16	1	4	16
0	0	0	0	0	2	8	32	2	8	32
0	0	0	0	1	4	16	1	4	16	1
0	0	0	0	2	8	32	2	8	32	2
0	0	0	1	4	16	1	4	16	1	4
0	0	0	2	8	32	2	8	32	2	8
0	0	1	4	16	1	4	16	1	4	16
0	0	2	8	0	2	8	0	2	8	0
0	1	4	0	1	4	0	1	4	0	1
0	2	0	0	2	0	0	2	0	0	2
1	0	0	1	0	0	1	0	0	1	0

The pattern is now apparent. The monoid $\mathcal{S}$ is generated by (3,0), (1,2), and (1,1) and is not normal because of the 0's in rows 0, 1, 2, 3 below the ray through $(1, 2)$. $\mathbb{Z}\mathcal{S}$ is the entire group $\mathbb{Z}^2$ so $\widetilde{\mathcal{S}}$ is all the integer points in the cone $j \geq 0, 2i \geq j$. Thus ${}^+\mathcal{S}$ is equal to $\widetilde{\mathcal{S}}$, except for the omission of the points on the t-axis which are not congruent to 0 mod 3. (If $\mathbf{M}$ is one of these omitted points and $m \equiv 1$ mod 3 then $m\mathbf{M} \notin \mathcal{S}$, so these points are not in ${}^+\mathcal{S}$. This shows the need to use relative interiors in the geometric description of ${}^+\mathcal{S}$.) There is only one prime 2, and $\mathcal{O} = \mathcal{O}_2$ is the submonoid of $\mathbb{Z}^2$ generated by (3,0) and (1,2). Therefore ${}^+A$ has the following table of coefficients:

0	0	0	0	0	1	2	2	1	2	2
0	0	0	0	0	2	2	2	2	2	2
0	0	0	0	1	2	2	1	2	2	1
0	0	0	0	2	2	2	2	2	2	2
0	0	0	1	2	2	1	2	2	1	2
0	0	0	2	2	2	2	2	2	2	2
0	0	1	2	2	1	2	2	1	2	2
0	0	2	2	2	2	2	2	2	2	2
0	1	2	2	1	2	2	1	2	2	1
0	2	2	2	2	2	2	2	2	2	2
1	0	0	1	0	0	1	0	0	1	0

and ${}^+A = \mathbb{Z}[t^3, tu^2, 2tu, 2t^2u, 2t^3u, 2t^2u^2, 2t^3u^2]$.

To find the weak normalization consider $\tau = \mathcal{C}(\mathcal{S})$, $q = 2$. The group $\mathbb{Z}({}^+\mathcal{O}_2 \cap \tau)$ has $\mathbb{Z}$-basis $\{(3,0),(1,2)\}$, which after a bit of row reduction becomes $\{(0,6),(1,2)\}$. A basis for the group $\mathbb{Z}({}^+\mathcal{S} \cap \tau) = \mathbb{Z}^2$ is $\{(0,1),(1,2)\}$. We remove the factor 2 from the 6 obtaining that ${}^*\mathcal{O}_2 \cap \mathring{\tau}$ is the intersection with $\mathring{\tau}$ of the subgroup with basis $\{(0,3),(1,2)\}$. On the two one-dimensional faces of τ we already have only 0's and 1's so nothing happens there. Hence the table of coefficients for ${}^*A$ is as follows:

0	0	0	0	0	1	2	2	1	2	2
0	0	0	0	0	2	1	2	2	1	2
0	0	0	0	1	2	2	1	2	2	1
0	0	0	0	2	1	2	2	1	2	2
0	0	0	1	2	2	1	2	2	1	2
0	0	0	2	1	2	2	1	2	2	1
0	0	1	2	2	1	2	2	1	2	2
0	0	2	1	2	2	1	2	2	1	2
0	1	2	2	1	2	2	1	2	2	1
0	2	1	2	2	1	2	2	1	2	2
1	0	0	1	0	0	1	0	0	1	0

and ${}^*A = \mathbb{Z}[t^3, tu^2, 2tu, t^2u, 2t^2u^2, 2t^3u]$. Another way to look at what we have done is to note that a set of representatives for the order 6 group $G_{2,\tau} = \mathbb{Z}({}^+\mathcal{S} \cap \tau)/\mathbb{Z}({}^+\mathcal{O}_2 \cap \tau)$ is (0,0), (1,0), (2,0), (1,1), (2,1), (3,1). Of these, only (2,1) is of order 2. We obtain the weak normalization ${}^*A$ by taking coefficient 1 for all translates of (2,1).

If $\mathbf{M} = (2,1)$ then $\mathcal{R}(\mathbf{M}) = \{\lambda(2,1) \mid \lambda \in \mathbb{N}\}$, and $\mathbb{Z}[\mathcal{R}(\mathbf{M})] = \mathbb{Z}[t^2u] \cong \mathbb{Z}[v]$ where $v = t^2u$. Then $A(\mathbf{M}) = \mathbb{Z} \oplus t^4u^2\mathbb{Z} \oplus 8t^6u^3\mathbb{Z} \oplus t^8u^4\mathbb{Z} \oplus \ldots = \mathbb{Z}[t^4u^2, 8t^6u^3]$, ${}^+(A(\mathbf{M})) = ({}^+A)(\mathbf{M}) = \mathbb{Z} \oplus 2t^2u\mathbb{Z} \oplus t^4u^2\mathbb{Z} \oplus \ldots = \mathbb{Z}[2t^2u, t^4u^2] \cong \mathbb{Z}[2v, v^2]$, and

${}^*(A(\mathbf{M})) = ({}^*A)(\mathbf{M}) = \mathbb{Z}[t^2u] \cong \mathbb{Z}[v]$. The coefficient sequences for A, ${}^+A$, and ${}^*A$ in this ray are respectively 1 0 1 8 1 8 1 . . . , 1 2 1 2 1 2 1 . . . , and 1 1 1 1 1 . . . respectively.

More generally the (coefficients in the) rays $\mathcal{R}(\mathbf{M})$ of ${}^*A$, for $\mathbf{M} \in \mathcal{S}$, $\mathbf{M} \neq (1,0)$ are all either 1 1 1 1 . . . or 1 2 2 1 2 2 1, ... The rays of ${}^+A$ are all either 1 1 1 1 1 . . . , 1 2 1 2 1 2 1 . . . , 1 2 2 1 2 2 1 2 2 1 . . . or 1 2 2 2 2 2 1 2 2 2 2 2 1 . . . . The rays of A are a bit more complicated. If a ray does not intersect rows one or two then the coefficients are one of 1 1 1 . . . , 1 2 4 8 16 32 1 . . . , 1 4 16 1 . . . , 1 8 1 . . . , 1 16 4 1 . . . , or 1 32 16 8 4 2 1 . . . (each repeating periodically and obtained by cycling through the (stable) coefficients of the representatives of $G_{2,7}$ according to the group law of $G_{2,7}$). If the ray meets rows one, two or three then the entries from these rows (except for 1,2, or 4) will be zero and the ray will stabilize at one of those listed above. In this example, as in the one-dimensional case, we can think of the construction of ${}^+A$ as "extending back to 0" by periodicity, and then taking the radical.

Next we determine the conductor of a submonoid $\mathcal{S}$ of $\mathbb{N}^n$ in its normalization $\widetilde{\mathcal{S}}$, where $\mathcal{S}$ is generated by $n+1$ elements. This generalizes the well known case of the postage stamp problem in the case $n = 1$, where the conductor in $\mathbb{N}$ of the submonoid generated by two relatively prime integers p and q is $pq - p - q + 1$. Given the difficulty of the postage stamp problem on more than two integers we do not expect a nice answer when $\mathcal{S}$ is generated by more than $n+1$ elements.

Theorem 5.2. *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n+1}$ be vectors in $\mathbb{N}^n$ such that any set of n of them is linearly independent, let $\mathcal{S}$ denote the submonoid of $\mathbb{N}^n$ they generate, $\overline{\mathcal{S}}$ its normalization (in the group $\mathbb{Z}\mathcal{S}$ generated by the $\mathbf{v}_i$), and let $\sigma = \mathcal{C}(\mathcal{S})$, the rational polyhedral cone generated by the $\mathbf{v}_i$. If d_i is the order of $\mathbf{v}_i$ in $\mathbb{Z}^n$ modulo the subgroup the group generated by $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots \mathbf{v}_n$, then the conductor of the extension $\mathcal{S} \subseteq \overline{\mathcal{S}}$ is $\lambda + (\mathring{\sigma} \cap \overline{\mathcal{S}})$, where $\lambda = (\sum_{i=1}^{n+1} d_i \mathbf{v}_i)/2 - \sum_{i=1}^{n+1} \mathbf{v}_i$ and $\mathring{\sigma}$ is the interior of σ.*

Proof. Write $\mathbf{v}_{n+1}$ as a (necessarily rational) linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The coefficient of $\mathbf{v}_i$ cannot be zero , since the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots \mathbf{v}_{n+1}\}$ is assumed to be linearly independent. Without loss of generality, assume that the coefficients of $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_r$ are positive and those of $\mathbf{v}_{r+1}, \ldots \mathbf{v}_n$ negative. Clearing denominators and rewriting, we obtain

5.2.1. $m_1\mathbf{v}_1 + \ldots + m_r\mathbf{v}_r = m_{r+1}\mathbf{v}_{r+1} + \ldots + m_{n+1}\mathbf{v}_{n+1}$

with $m_i \in \mathbb{Z}_+$ and $\mathrm{GCD}(\{m_i\}) = 1$. (We must have at least one term on each side of the equation (i.e. $1 \leq r \leq n$); if not, $\mathbf{v}_{n+1}$ would be invertible in $\mathcal{S}$, contradicting the fact that $\mathcal{S} \subseteq \mathbb{N}^n$. One may also assume, without loss of generality, that $r \leq \lfloor (n+1)/2 \rfloor$, but we will not need this.) We claim that $m_i = d_i$. It is clear that $d_i \mid m_i$, so let $\alpha_i = m_i/d_i$. We are given that $d_i\mathbf{v}_i = \beta_1\mathbf{v}_1 + \ldots + \beta_{i-1}\mathbf{v}_{i-1} + \beta_{i+1}\mathbf{v}_{i+1} + \ldots \beta_{n+1}\mathbf{v}_{n+1}$, $\beta_j \in \mathbb{Z}$. Multiplying by α_i and applying the fact that any equation of linear dependence involving the $\mathbf{v}_i$ must be a multiple of 5.2.1, we conclude that $m_j = \pm\beta_j\alpha_i$ $(j \neq i)$. But this implies that $\mathrm{GCD}(\{m_j\}) \geq \alpha_i$, so $\alpha_i = 1$ and $m_i = d_i$. We now have $\Sigma_{i=1}^r d_i\mathbf{v}_i = \Sigma_{i=r+1}^{n+1} d_i\mathbf{v}_i$ from which it follows that

5.2.2. $\lambda = \sum_{i=1}^r d_i\mathbf{v}_i - \sum_{i=1}^{n+1} \mathbf{v}_i$

and indeed $\lambda \in \mathbb{Z}\mathcal{S}$. We must show that $\lambda + (\sigma^\circ \cap \overline{\mathcal{S}}) \subseteq \mathcal{S}$ (from which it is immediate that $\lambda + (\sigma^\circ \cap \overline{\mathcal{S}}) + \overline{\mathcal{S}} \subseteq \mathcal{S}$ so $\lambda + (\sigma^\circ \cap \overline{\mathcal{S}})$ is contained in the conductor) and that if $\mathbf{x} \in \mathcal{S}\backslash(\lambda + (\sigma^\circ \cap \overline{\mathcal{S}}))$ then there is an $\mathbf{s} \in \overline{\mathcal{S}}$ such that $\mathbf{x} + \mathbf{s} \notin \mathcal{S}$ (so that nothing else is contained in the conductor). Note that if $(a_1, a_2, \ldots, a_{n+1}) \in \mathbb{Z}^{n+1}$ is one solution to $\sum_{i=1}^{n+1} x_i \mathbf{v}_i = \mathbf{w}$, then all integer solutions are given by $(a_1, a_2, \ldots, a_{n+1}) + t(d_1, \ldots, d_r, -d_{r+1}, \ldots, -d_{n+1})$ (the fact that $\mathrm{GCD}(\{d_i\}) = 1$ is used here).

We will now show that $\lambda + (\mathring{\sigma} \cap \overline{\mathcal{S}}) \subseteq \mathcal{S}$. Note that any $\mathbf{u} \in \mathring{\sigma} \cap \overline{\mathcal{S}}$ can be written as $\mathbf{u} = \sum_{i=1}^{n+1} \alpha_i \mathbf{v}_i$ with $\alpha_i \in \mathbb{Q}_+$ $(1 \leq i \leq n+1)$. We must show that we can solve

5.2.3. $d_1\mathbf{v}_1 + \ldots + d_r\mathbf{v}_r - \sum_{i=1}^{n+1} \mathbf{v}_i + \mathbf{u} = \sum_{i=1}^{n+1} x_i \mathbf{v}_i$, with $x_i \in \mathbb{N}$.

The set of all real solutions to 5.2.3 is a line parallel to the vector $(d_1, \ldots, d_r, -d_{r+1}, \ldots, -d_{n+1})$. Now $\mathbf{w}_1 = (d_1 + \alpha_1 - 1, ..., d_r + \alpha_r - 1, \alpha_{r+1} - 1, \ldots, \alpha_{n+1} - 1)$ and $\mathbf{w}_2 = (\alpha_1 - 1, ..., \alpha_r - 1, d_{r+1} + \alpha_{r+1} - 1, \ldots, d_{n+1} + \alpha_{n+1} - 1)$ are two (rational) solutions to 5.2.3, so they give two points on the line of real solutions. Since $\mathbf{w}_1 - \mathbf{w}_2 = (d_1, \ldots, d_r, -d_{r+1}, \ldots, -d_{n+1})$, there must be an integer solution $\mathbf{b} = (b_1, \ldots, b_{n+1})$ between them (by the description of all integer solutions given in the previous paragraph). But since $\mathbf{b}$ is between $\mathbf{w}_1$ and $\mathbf{w}_2$, we have $-1 < \alpha_i - 1 \leq b_i \leq d_i + \alpha_i - 1$, so in particular $b_i \geq 0$.

To finish, we must prove a series of intermediate results. First, we claim

5.2.4. $\sigma\backslash\mathring{\sigma} = \{\mathbf{t} \mid \mathbf{t} = \sum_{i=1}^{n+1} \alpha_i \mathbf{v}_i$ with $\alpha_i \in \mathbb{R}_{\geq 0}$, and for at least one $j, 1 \leq j \leq r$, and at least one $k,\ r+1 \leq k \leq n+1,\ \alpha_j = \alpha_k = 0\}$.

If $\mathbf{t}$ is an element of the set on the right-hand side of the alleged equality, then $\mathbf{t}$ is clearly in σ. If $\mathbf{t}$ were also in σ°, then we could write $\mathbf{t} = \sum_{i=1}^{n+1} \beta_i \mathbf{v}_i$ with $\beta_i > 0$ $(1 \leq i \leq n+1)$. Without loss of generality, we may take $j = r$ and $k = n+1$. Equating the two expressions for $\mathbf{t}$ and rewriting, we obtain $\sum_{i=1}^{r-1} (\alpha_i - \beta_i)\mathbf{v}_i - \beta_r \mathbf{v}_r = \sum_{i=r+1}^{n} (\beta_i - \alpha_i)\mathbf{v}_i + \beta_{n+1}\mathbf{v}_{n+1}$. But any relation among the $\mathbf{v}_i$ must be a multiple of 5.2.1 whose coefficients are all positive. The fact that $-\beta_r$ and β_{n+1} have opposite sides gives a contradiction, hence $\mathbf{t} \in \sigma\backslash\sigma^\circ$. On the other hand, if $\mathbf{t}$ is not an element of the set on the right-hand side of the alleged equality, then either $\alpha_i > 0$ for $1 \leq i \leq r$ and $\alpha_i \geq 0, r+1 \leq i \leq n+1$, or $\alpha_i > 0$ for $r+1 \leq i \leq n+1$ and $\alpha_i \geq 0,\ 1 \leq i \leq r$. In the first case let $\epsilon = \min\{\alpha_i/d_i \mid i = 1, \ldots, r\}/2$, then $\mathbf{t} = \sum_{i=1}^{r} (\alpha_i - \epsilon d_i)\mathbf{v}_i + \sum_{i=r+1}^{n+1} (\alpha_i + \epsilon d_i)\mathbf{v}_i$ has all coefficients positive so $t \in \mathring{\sigma}$ and we are done. The second case is similar.

Our second claim is

5.2.5. if $\mathbf{t} \in \mathcal{S} \cap (\sigma\backslash\mathring{\sigma})$, then $\mathbf{t} = \sum_{i=1}^{n+1} c_i \mathbf{v}_i$ with $c_i \in \mathbb{N}$, and for at least one $j,\ 1 \leq j \leq r$, and at least one $k,\ r+1 \leq k \leq n+1,\ c_j = c_k = 0$.

In other words 5.2.5 shows that the α_i in 5.2.4 may be taken to be in $\mathbb{N}$. We are given that $\mathbf{t} = \sum_{i=1}^{n+1} c_i \mathbf{v}_i$ with $c_i \in \mathbb{N}$ and by 5.2.4, $\mathbf{t} = \sum_{i=1}^{r-1} \alpha_i \mathbf{v}_i + \sum_{i=r+1}^{n} \alpha_i \mathbf{v}_i$ (we have again taken $j = r,\ k = n+1$ without loss of generality). Equating these two expressions for $\mathbf{t}$ and rewriting, we obtain $\sum_{i=1}^{r-1} (c_i - \alpha_i)\mathbf{v}_i + c_r \mathbf{v}_r = \sum_{i=r+1}^{n} (\alpha_i - c_i)\mathbf{v}_i - c_{n+1}\mathbf{v}_{n+1}$. But, as in the proof of 5.2.4, c_r and $-c_{n+1}$ cannot have opposite signs and hence must both be zero. Since any n of the $\{\mathbf{v}_i\}$ are linearly independent, all of the coefficients must be zero, and 5.2.5 follows. Finally we claim

5.2.6. if $\mathbf{u} \in \lambda + (\mathcal{S} \cap (\sigma \backslash \sigma^\circ))$, then $\mathbf{u} \notin \mathcal{S}$.

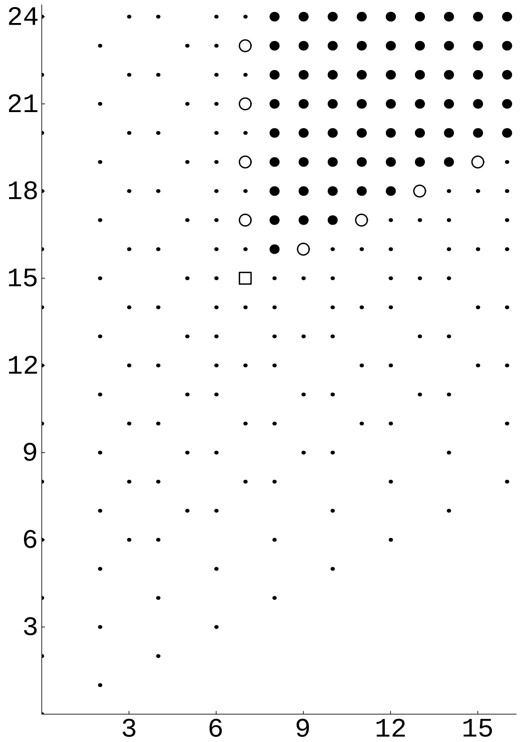
Let $\mathbf{u} \in \lambda + (\mathcal{S} \cap (\sigma \backslash \sigma^\circ))$. We must show that if $\mathbf{u} = \sum_{i=1}^{n+1} x_i \mathbf{v}_i$ $(x_i \in \mathbb{R})$, then at least one of the x_i is negative. From the definition of λ and 5.2.5, $\mathbf{u} = \sum_{i=1}^{r} d_i \mathbf{v}_i - \sum_{i=1}^{n+1} \mathbf{v}_i + \sum_{i=1}^{n+1} c_i \mathbf{v}_i$ where all coefficients are integers and we may assume without loss of generality that $c_r = c_{n+1} = 0$. This gives an integer solution $(a_1, \dots, a_{n+1})$ to $\mathbf{u} = \sum_{i=1}^{n+1} x_i \mathbf{v}_i$ with $a_r = d_r - 1$ and $a_{n+1} = -1$. As remarked above, all integer solutions to this equation are therefore given by $(a_1, \dots, a_{n+1}) + t(d_1, \dots, d_r, -d_{r+1}, \dots, -d_{n+1}), t \in \mathbb{Z}$. But if $t \geq 0$, then x_{n+1} will be negative and if $t < 0$, then x_r will be negative, so there are no non-negative solutions to $\mathbf{u} = \sum_{i=1}^{n+1} x_i \mathbf{v}_i$, hence $\mathbf{u} \notin \mathcal{S}$, completing the proof of 5.2.6.

Finally, suppose that $\mathbf{x} \in \mathcal{S} \backslash (\lambda + (\mathring{\sigma} \cap \overline{\mathcal{S}}))$. We wish to find an $\mathbf{s} \in \overline{\mathcal{S}}$ such that $\mathbf{x} + \mathbf{s} \notin \mathcal{S}$, which will complete the proof of Theorem 5.2. Write $\mathbf{x} - \lambda = \sum_{i=1}^{n+1} \alpha_i \mathbf{v}_i$ with $\alpha_i \in \mathbb{Q}$. We will show that we may assume there is at least one j, $1 \leq j \leq r$, and at least one k, $r+1 \leq k \leq n+1$, such that $\alpha_j \leq 0$ and $\alpha_k \leq 0$. For any rational β_i such that $\mathbf{x} - \lambda = \sum_{i=1}^{n+1} \beta_i \mathbf{v}_i$, we must have $\beta_i = \alpha_i + t d_i$, $1 \leq i \leq r$, and $\beta_i = \alpha_i - t d_i$, $r+1 \leq i \leq n+1$. Let $p = \max(\{-\alpha_i / d_i\}_{i=1,..,r}) = -\alpha_j / d_j$ and $q = \min(\{\alpha_i / d_i\}_{i=r+1,..,n+1}) = \alpha_k / d_k$. For all $t > p$, $\beta_i > 0$, $1 \leq i \leq r$ and for all $t < q$, $\beta_i > 0$, $r+1 \leq i \leq n+1$. Since we are assuming that $\mathbf{x} - \lambda \notin \sigma^\circ$, it cannot be the case that all of the β_i are positive, therefore we must have $q \leq p$. Choosing t so that $q \leq t \leq p$ and taking the corresponding β_i, we have $\beta_j \leq 0$ and $\beta_k \leq 0$. We replace the original α_i with the β_i to finish the proof of the claim.

Without loss of generality, assume that $\mathbf{x} - \lambda = \sum_{i=1}^{n+1} \alpha_i \mathbf{v}_i$ with $\alpha_i \in \mathbb{Q}$, $\alpha_i \leq 0$ for $1 \leq i \leq j$, $\alpha_i > 0$ for $j+1 \leq i \leq r$, $\alpha_i \leq 0$ for $r+1 \leq i \leq k$, and $\alpha_i > 0$ for $k+1 \leq i \leq n+1$. Let $\mathbf{s} = -\mathbf{x} + \lambda + \sum_{i=j+1}^{r} \lceil \alpha_i \rceil \mathbf{v}_i + \sum_{i=r+1}^{k} \lceil \alpha_i \rceil \mathbf{v}_i = \sum_{i=1}^{j} (-\alpha_i) \mathbf{v}_i + \sum_{i=j+1}^{r} (\lceil \alpha_i \rceil - \alpha_i) \mathbf{v}_i + \sum_{i=r+1}^{k} (-\alpha_i) \mathbf{v}_i + \sum_{i=k+1}^{n+1} (\lceil \alpha_i \rceil - \alpha_i) \mathbf{v}_i$. The first expression shows that $\mathbf{s}$ is in $\mathbb{Z}\mathcal{S}$ and the second shows that $\mathbf{s} \in \sigma$, therefore $\mathbf{s} \in \overline{\mathcal{S}}$. But $\mathbf{x} + \mathbf{s} = \lambda + \sum_{i=j+1}^{r} \lceil \alpha_i \rceil \mathbf{v}_i + \sum_{i=k+1}^{n+1} \lceil \alpha_i \rceil \mathbf{v}_i \in \lambda + (\mathcal{S} \cap (\sigma \backslash \mathring{\sigma}))$ so $\mathbf{x} + \mathbf{s} \notin \mathcal{S}$ by 5.2.6. This completes the proof of Theorem 5.2. $\square$

Example 5.3. Take $n = 2$, $\mathbf{v}_1 = (2,1)$, $\mathbf{v}_2 = (0,2)$, $\mathbf{v}_3 = (3,6)$. Here $\det(\mathbf{v}_1, \mathbf{v}_2) = \det \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} = 4$ and $\mathbb{Z}^2 / \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \cong \mathbb{Z}/4\mathbb{Z}$. The order of $\mathbf{v}_3$ in $\mathbb{Z}^2 / \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ is 4 so $d_3 = 4$. Similarly $d_1 = 6, d_2 = 9$ and $\lambda = (6(2,1) + 9(0,2) + 4(3,6))/2 - (2,1) - (0,2) - (3,6) = (7,15)$.

In the picture below the large dots are the elements of the conductor, λ is the open square, the remaining elements of $\{\lambda + (\mathcal{S} \cap (\sigma \backslash \mathring{\sigma})\}$ are the open circles. These are not in $\mathcal{S}$ by 5.2.6. The small dots are the elements of $\mathcal{S}$ that are not in the conductor. The final step in the proof of 5.2 is to show that for every small dot $\mathbf{x}$ there is an element $\mathbf{s} \in \overline{\mathcal{S}}$ such that $\mathbf{x} + \mathbf{s}$ is one of the open circles. Since $\mathcal{S}$ is the cone spanned by $(2,1)$ and $(0,2)$ this seems reasonably obvious here, and more generally in the two-dimensional case. The higher-dimensional case is complicated by the fact that cones need no longer have simplicial cross sections.



In the above example, $\lambda \in \mathcal{C}(\mathcal{S})$. However in general λ need not lie in $\mathcal{C}(\mathcal{S})$, as is illustrated by the example $\mathbf{v}_1 = (5,2), \mathbf{v}_2 = (2,3), \mathbf{v}_3 = (1,1)$, where $\lambda = (3,5)$ and $\mathcal{C}(\mathcal{S})$ is spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

6. The case of not enough 1's.

In this section we make a few remarks about the case where there are "not enough ones". By Corollary 3.4 it suffices to consider the case $A = A^{[q]}$ for a prime q. Suppose that $A = \mathbb{Z}[\{a_{\mathbf{M}}\mathbf{t}^{\mathbf{M}}\}], a_{\mathbf{M}} = q^{e_{\mathbf{M}}}$, where the set of algebra generators need not be finite. Let $B = \mathbb{Z}[u, \{u^{e_{\mathbf{M}}}\mathbf{t}^{\mathbf{M}}\}]$. That is, replace q by an extra indeterminate u. Corresponding to the fact that an element of A can be multiplied by q by adding it q times we also add u to the indeterminates. Then we have that if $T = q^{a_{\mathbf{M}}}\mathbf{t}^{\mathbf{M}}$ then $T^n \in A$ if and only if $(u^{a_{\mathbf{M}}}\mathbf{t}^{\mathbf{M}})^n \in B$. Therefore, by Theorem 3.2 (and Lemma 3.1!) we obtain

Theorem 6.1. *With the notation above, A is seminormal (respectively normal) if and only if B is seminormal (respectively normal). The ring B is the monoid ring of a monoid $\mathcal{T} \subseteq \mathbb{R}^{n+1}$, and by Theorem 3.8, B is normal (respectively seminormal) if and only if $\mathcal{T}$ has the corresponding property. And $\mathcal{T}$ is normal (respectively seminormal) if and only if it satisfies Remark 3.7 (respectively Theorems 4.3 or 4.18).*

Cones can be very complicated in $\mathbb{R}^i, i \geq 3$, and the possibility that A might not be a finitely generated $\mathbb{Z}$-algebra complicates things further, so we can hope for a truly explicit general solution only if $n = 1$ where we have:

Theorem 6.2. *The seminormal monomial subrings $\mathbb{Z}[\{a_i t^i\}_{i \geq 1}]$ ($a_i = q^{e_i}$ for some prime q) of $\mathbb{Z}[t]$ with $\mathbb{Z}\mathcal{S} = \mathbb{Z}$ are one of the following:*

(a) *for some* $\lambda \in \mathbb{R}\backslash\mathbb{Q}, \lambda > 0, e_i = \lceil \lambda i \rceil$. *(i.e.* e_i *is the e-coordinate of the point* (i, e_i) *with integer co-ordinates which lies just above the line* $e = \lambda i$.*)*
(b) *for some* $\lambda \in \mathbb{Q},\ \lambda \geq 0, e_i = 1 + \lfloor \lambda i \rfloor$. *(i.e.* e_i *is the e-coordinate of the point* (i, e_i) *with integer co-ordinates which lies just above the line* $e = \lambda i$.*)*
(c) *for some* $\lambda = m/n \in \mathbb{Q}, m, n$ *relatively prime integers, and some integer* $d \geq 1$, *then* $e_i = 1 + \lfloor \lambda i \rfloor$ *except if* i *is an integer multiple* cnd *of* nd, *in which case* $e_i = \lambda i = (m/n)(cnd) = mcd$. *(i.e. to the points of* (b) *we add every* d^{th} *integer point on the line* $e = \lambda i$*).*

Cases (a) *and* (b) *are not finitely generated as a* $\mathbb{Z}$*-algebra, and are normal by Remark 3.7. Case* (c) *is finitely generated as a* $\mathbb{Z}$*-algebra, and is not normal unless* $d = 1$. *Case* (c) *with* $\lambda = 0$ *is the case previously studied in this paper (the case* $n = 1$ *of Theorem* 4.10*) and in* [12].

Proof. As discussed above, let u be a second indeterminate and let $B = \mathbb{Z}[u, \{u^{e_i}t^i\}]$. Also as discussed above, A is seminormal if and only if the monoid associated to B is seminormal. This reduces us to finding all seminormal submonoids $\mathcal{T}$ of $\mathbb{N}^2$ which contain (0,1) and for which $\mathbb{Z}\mathcal{T} = \mathbb{Z}^2$. The cone $\mathcal{C}(\mathcal{T})$ is either closed or open lying between the (0,1) axis and the half-line $(c, c\lambda)$ with slope λ $(0 \leq \lambda < 1)$ and $c \geq 0$. If λ is not rational there are no points of $\mathcal{T}$ on this half-line. Suppose first that either λ is not rational or that λ is rational and there are no points of $\mathcal{T}$ on the line. For any rational number μ $(\lambda < \mu < 1)$ let $\mathcal{C}_\mu$ be the cone between the half-line through (0,1) and the half-line $(c, \mu c)$ (including these half-lines). Then $\mathcal{C}_\mu \cap \mathcal{T}$ is a finitely generated monoid by Corollary 4.8. If we apply Theorem 4.3 to $\mathcal{C}_{\mu'}$ where μ' is rational, $\lambda < \mu' < \mu$ and let μ' approach λ from above we conclude that $\mathcal{C}_\lambda \cap \mathcal{T} = \mathcal{C}_\lambda \cap \mathbb{Z}^2$ and we have either case (a) or (b). If λ is rational and there are points of $\mathcal{T}$ on the line $(c, c\lambda)$ then we have case (c) by Theorem 4.3. □

Remark 6.3. The weakly normal subrings of $\mathbb{Z}[\{a_it^i\}_{i\geq 1}]$ $(a_i = q^{e_i}$ for some prime $q)$ of $\mathbb{Z}[t]$ with $\mathbb{Z}\mathcal{S} = \mathbb{Z}$ will be cases (a) and (b) of Theorem 6.2 (they are weakly normal since they are normal) and case (c) with the d'^{th} point on the line $e = \lambda i$ (d' the maximal factor of d that is relatively prime to q).

References

[1] D. D. Anderson and D. F. Anderson. Divisorial ideals and invertible ideals in a graded domain. *J.Algebra*, 76:549–569, 1982.
[2] D. Avis. User's guide for lrs - version 3.2a. available from lrs homepage ftp://mutt.ca.mcgill.ca/pub/C/lrs.html, 1998.
[3] N. Bourbaki. *Algèbre Commutative*, chapter 5, Entiers. Hermann, 1964.
[4] T. Christof and A.Loebel. PORTA, a POlyhedron Representation Transformation Algorithm (version 1.3.2). available from http://www.iwr.uni-heidelberg.de/iwr/comopt/soft/PORTA/readme.html, 1998.
[5] A. Conca, J.Herzog, N.V.Trung, and G.Valla. Diagonal subalgebras of bigraded algebras and embeddings of blow-ups of projective spaces. *Amer. J. Math.*, 119:296–314, 1997.
[6] K. Fukuda. cdd (version 061a). Program and Reference Manual available from http://www.ifor.math.ethz.ch/ifor/staff/fukuda/cddhome/cdd.html, 1997.
[7] R. Gilmer. *Commutative Semigroup Rings*. The University of Chicago Press, 1984.
[8] J. Gubeladze. Anderson's conjecture and the maximal monoid class over which projective modules are free. *Math. USSR Sbornik*, 63:165–180, 1989.
[9] J. Gubeladze. Geometric and algebraic representations of commutative cancellative monoids. *Proc. of A. Razmadze Math. Inst.*, 113:31–81, 1995.
[10] J. Gubeladze. Nontriviality of $SK_1(R[M])$. *JPAA*, 104:169–190, 1995.
[11] Les Reid and M.Vitulli. The weak subintegral closure of a monomial ideal. To appear in *Comm.Alg.*

[12] Les Reid and Leslie G. Roberts. Monomial subrings and systems of subintegrality. To appear in *JPAA*.

[13] Les Reid and Leslie G. Roberts. A new criterion for weak subintegrality. *Comm.Alg.*, 24(10):3335–3342, 1996.

[14] Les Reid, Leslie G. Roberts, and Balwant Singh. On weak subintegrality. *JPAA*, 114:93–109, 1996.

[15] L. Robbiano and G. Valla. Hilbert-Poincaré series of bigraded algebras. *Boll. Un. Mat. Ital.*, 8(1-B):541–570, 1998.

[16] R. G. Swan. On seminormality. *J.Algebra*, 67:210–229, 1980.

[17] R. G. Swan. Gubeladze's proof of Anderson's conjecture. In *Azumaya Algebras, Actions, and Modules*, number 124 in Contemporary Mathematics, pages 215–250. American Mathemtical Society, 1992.

[18] H. Yanagihara. On an intrinsic definition of weakly normal rings. *Kobe J. Math.*, 2:89–98, 1985.

[19] G. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, 1995.

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