Part 3 of Tony Geramita's Fall 1994 Homological Algebra Notes

sect10a.tex

If we apply the ideas above to the right exact functor

$$- \otimes_R M : R - \text{mod} \to R - \text{mod}$$

for a fixed R-module M, we get its left-derived functors. Instead of using the "generic" notation, $L_n F$, in this case we call the left derived functors, $Tor_n^R(-, M)$.

Why the name "Tor? The following example will give some indication of why that name was used.

Proposition: Let B be a fixed $\mathbb{Z}$-module, i.e. an abelian group, and consider the abelian groups, $\mathbb{Z}/n\mathbb{Z}$. Then

$$Tor_i^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) = \begin{cases} B/nB, & \text{if } i = 0 \text{ ;} \\ {}_nB, & \text{if } i = 1 \text{ ;} \\ 0 & \text{if } i \geq 2 \end{cases}$$

i.e. $Tor_1^{\mathbb{Z}}(-, B)$ calculates the n-torsion subgroup of B when it is evaluated on the abelian group $\mathbb{Z}/n\mathbb{Z}$.

Proof: In order to make these calculations, we need to get a projective resolution of the abelian group $\mathbb{Z}/n\mathbb{Z}$. The following works:

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

i.e. a projective resolution of $\mathbb{Z}/n\mathbb{Z}$ is

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to 0$$

If we apply the functor $- \otimes_{\mathbb{Z}} B$ to this, we get the complex

$$\cdots 0 \to \mathbb{Z} \otimes_{\mathbb{Z}} B \xrightarrow{\times n} \mathbb{Z} \otimes_{\mathbb{Z}} B \to 0$$

which becomes

$$\cdots \to 0 \to B \xrightarrow{\times n} B \to 0$$

The 0-th homology of this complex is B/nB, and the 1st homology is the n-torsion of the group B while the higher homology groups are clearly 0. $\square$

There is also another collection of right exact functors associated with the tensor product, namely the functors

$$M \otimes_R - : R - \text{mod} \ \to R - \text{mod} \ .$$

It is traditional to refer to the left derived functors of this functor as $Tor_i^R(M, -)$.

But, this immediately creates a problem! What does $Tor_i^R(M, N)$ mean? Is it $Tor_i^R(M, -)$ evaluated at the R-module N? or is it $Tor_i^R(-, N)$ evaluated at the R-module M?

As you might hope (and guess) it doesn't matter!! But, this is quite a complicated theorem to prove and would take us a bit of time, which I don't want to spend on it (important as the fact is!). I'll forego the proof this term and just refer you to Weibel's book (for example) for a proof. It follows from this that

$$Tor_i^R(A, B) \simeq Tor_i^R(B, A)$$

when R is a commutative ring.

Once we have observed this we have

Corollary: If P is a projective R-module then

$$Tor_i^R(P, -) = Tor_i^R(-, P) = 0 \text{ for } i \geq 1 \ .$$

Exercise 19A:

If $(P_{i,\bullet}, d_{i,\bullet})$ are projective resolutions of the modules (A_i) then $(\oplus P_{i,\bullet}, \oplus d_{i,\bullet})$ is a projective resolution of $\oplus A_i$. Prove this and show how that implies that $(L_nF)(\oplus A_i) \simeq \oplus(L_nF)(A_i)$.

Corollary: Let A be a finitely generated abelian group and let B be any group. Then,

i) $Tor_i^{\mathbb{Z}}(A, B)$ is a torsion abelian subgroup of a finite direct sum of copies of B;

ii) $Tor_i^{\mathbb{Z}}(A,B) = 0$ for $i \geq 2$.

In particular, if B is a torsion-free abelian group and A is finitely generated then $Tor_i^{\mathbb{Z}}(A,B) = 0$ for $i \geq 1$.

Proof: We can write $A \simeq \mathbb{Z}^m \oplus \mathbb{Z}/(n_1) \oplus \cdots \oplus \mathbb{Z}/(n_r)$. Then

$$Tor_i^{\mathbb{Z}}(A,B) = Tor_i^{\mathbb{Z}}(\mathbb{Z}^m, B) \oplus \sum_{j=1}^{r} Tor_j^{\mathbb{Z}}(\mathbb{Z}/(n_j), B)$$

and the result follows from our earlier computations. $\square$

There is a slight imbalance in this proposition, and that's not a result of the facts of the case, but of something I have not proved about Tor_i^R. It is an additional feature of the two families of functors $- \otimes_R M$ and $M \otimes_R -$ that they (and their left derived functors) distribute over direct limits. I don't want to go into that right now either. But, if I did, the proposition above could be stated for ANY abelian group. In fact, we would have:

Proposition: The functor $Tor_i^{\mathbb{Z}}(A,-)$, $i \geq 1$, is identically zero if and only if A is a torsion free abelian group and $Tor_i^{\mathbb{Z}}(A,-) = 0$ for all $i \geq 2$ and any abelian group A.

Of course, the same statement is true for the functors $Tor_i^{\mathbb{Z}}(-,A)$.

It is worth remarking that the proposition above says a great deal about the ring $\mathbb{Z}$, as much as it says anything else! It is not true for any arbitrary commutative ring. I.e. we can find a commutative ring R and a finitely generated module A for which $Tor_i^R(A,-) \neq 0$ for any $i \geq 2$. The following example illustrates this.

Example: Let $R = \mathbb{Z}/(m)$ and suppose we have an integer d, such that $d|m$ and $1 < d < m$. Let $t = m/d$ and $M = \mathbb{Z}/(d)$. Then M is an R-module.

The following (infinite) periodic sequence is an exact complex:

$$\cdots \xrightarrow{\times d} \mathbb{Z}/(m) \xrightarrow{\times t} \mathbb{Z}/(m) \xrightarrow{\times d} \mathbb{Z}/(m) \xrightarrow{\epsilon} \mathbb{Z}/(d) \to 0$$

where ϵ is the quotient map of $(d)/(m) \subseteq \mathbb{Z}/(m)$.

Proof: Clearly $ker(\epsilon) = (d)/(m)$ and that is equally clearly the image of $\times d$.

Now $Ker(\times d) = \{r + (m) \mid dr = lm\}$. But $dr = lm \Rightarrow dr = l(dt)$ i.e. $r = tl$ and so $r \in (t)$ i.e. $Ker(\times d) \subseteq Im(\times t)$ and the other inclusion is clear.

Returning to this example, we know that if B is any $\mathbb{Z}/(m)$-module then, to calculate $Tor_i^{\mathbb{Z}/(m)}(M, B)$, we take any projective resolution of M, tensor it (over $\mathbb{Z}/(m)$) with B and then calculate homology. (Notice that B, being an abelian group, is a $\mathbb{Z}$-module. But, since it is a $\mathbb{Z}/(m)$-module also, we have that m acts trivially on B. This observation is valid for any B-module.)

So, we look at the complex above, tensor it with B and calculate homology, we get:

$$\cdots \xrightarrow{d\otimes id} \mathbb{Z}/(m) \otimes_{\mathbb{Z}/(m)} B \xrightarrow{t\otimes id} \mathbb{Z}/(m) \otimes_{\mathbb{Z}/(m)} B \xrightarrow{d\otimes id} \mathbb{Z}/(m) \otimes_{\mathbb{Z}/(m)} B \to 0$$

and calculate homology.

But recall that $R \otimes_R B \simeq B$ via $r \otimes b \to rb$. So, consider

$$\begin{array}{ccc} \mathbb{Z}/(m) \otimes_{\mathbb{Z}/(m)} B & \xrightarrow{d\otimes id} & \mathbb{Z}/(m) \otimes_{\mathbb{Z}/(m)} B \\ \chi \downarrow & & \chi \downarrow \\ B & \xrightarrow{\phi} & B \end{array} .$$

What is the map ϕ? Let $b \in B$ then (on the lefthand side) $b = \chi(\overline{1} \otimes b)$ and

$$(d \otimes id)(\overline{1} \otimes b) = \overline{d} \otimes b \text{ and } \chi(\overline{d} \otimes b) = \overline{d}b$$

(since m acts trivially on B). Thus

$$\phi(b) = db \ .$$

So, the complex above becomes

$$\cdots \xrightarrow{d} B \xrightarrow{t} B \xrightarrow{d} B \to 0 \ .$$

Thus we can compute

$$Tor_0^{\mathbb{Z}/(m)}(\mathbb{Z}/(d), B) = B/dB$$

$$Tor_1^{\mathbb{Z}/(m)}(\mathbb{Z}/(d), B) = \{b \mid db = 0\}/tB$$

and

$$Tor_2^{\mathbb{Z}/(m)}(\mathbb{Z}/(d), B) = \{b \mid tb = 0\}/dB$$

More generally,

$$Tor_n^{\mathbb{Z}/(m)}(\mathbb{Z}/(d), B) = \begin{cases} Tor_1^{\mathbb{Z}/(m)}(\mathbb{Z}/(d), B) & \text{if } n > 1,\ n \text{ odd ;} \\ Tor_2^{\mathbb{Z}/(m)}(\mathbb{Z}/(d), B) & \text{if } n > 1,\ n \text{ even.} \end{cases}$$

In particular, if we let $m = 12$, $d = 2$, $t = 6$ we see that the Tor_i^R are not zero for $i > 1$ and for this R.

The following observation will be very useful. It can be made quite generally.

Let M be an R-module and let F be a right exact functor. Let $r \in R$ and

$$\phi_{r,M} : M \to M \text{ be given by } \phi_{r,M}(m) = rm\ .$$

Lemma:

$$(L_iF)(\phi_{r,M}) : (L_iF)(M) \to (L_iF)(M) \text{ is } \phi_{r,(L_iF)(M)}$$

i.e. the induced map is again multiplication by r.

Proof: Notice that if $(P_\bullet, d_\bullet)$ is a projective resolution of M then the following diagram has each square commuting:

$$\begin{array}{ccccccccc} \cdots & \to & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M & \to & 0 \\ & & \phi_{r,P_1}\downarrow & & \phi_{r,P_0}\downarrow & & \downarrow\phi_r & & \\ \cdots & \to & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M & \to & 0 \end{array}$$

and that is enough to prove the Lemma since the homology modules are simply quotients of submodules of the modules in the squares. □

Exercise 19B: Let I and J be ideals in the commutative ring R. Then

$$Tor_1^R(R/I, R/J) \simeq (I \cap J)/IJ\ .$$

so this particular Tor can be used to measure how far product and intersection differ in a commutative ring R.

Hint: Show that the following is a commutative diagram, and then use it!

$$\begin{array}{ccccccccc} 0 & \to & IJ & \to & I & \to & I \otimes_R R/J & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J & \to & R & \to & R \otimes_R R/J & \to & 0 \end{array}$$

In particular, $Tor_1^R(R/I, R/I) \simeq I/I^2$.

Recall that we had defined, earlier, that M is flat if and only if $M \otimes_R -$ is an exact functor. From what we showed earlier,

$$M \text{ a flat } R-\text{module } \Rightarrow Tor_i^R(M, -) = Tor_i^R(-, M) = 0 \text{ for all } i > 0 \ .$$

In fact, the converse is also true! We have the following:

Proposition: TFAE

1) M is a flat R-module;
2) $Tor_i^R(M, -) = 0$ for all $i \geq 1$;
3) $Tor_1^R(M, -) = 0$.

Proof: 1) implies 2) from above and 2) implies 3) is obvious. So, suppose we are given 3). We need to show that $M \otimes_R -$ is an exact functor.

Consider a short exact sequence of R-modules,

$$0 \to A' \to A \to A'' \to 0 \ .$$

We have

$$\cdots Tor_1^R(M, A'') \to Tor_0^R(M, A') \to Tor_0^R(M, A) \to Tor_0^R(M, A'') \to 0 \ .$$

Since $Tor_1^R(M, A'') = 0$ and $Tor_0^R(M, B) \simeq M \otimes_R B$, we are done. □

Another very useful property of the derived functors for the tensor product comes from the following:

Proposition:(Flat base change for Tor)
Let $R \to T$ be a homomorphism of commutative rings which makes T into a flat R-module and let C be an $R-T$-bimodule. Then

$$Tor_n^R(A, C) \simeq Tor_n^T(A \otimes_R T, C) \ .$$

Proof: Let $P_\bullet \xrightarrow{\epsilon} A$ be a projective resolution of A. Then $Tor_n^R(A, C)$ is the n-th homology module of the complex $P_\bullet \otimes C$.

Now $P_n \otimes_R T$ is a projective T-module but, since T is R-flat (and so $- \otimes_R T$ is an exact functor) we get that $P_\bullet \otimes_R T$ is a projective resolution of $A \otimes_R T$.

Thus, we can calculate $Tor_n^T(A \otimes_R T, C)$ as the homology of the complex

$$P_\bullet \otimes_R T \otimes_T C \simeq P_\bullet \otimes_R C \ ,$$

and so we are done. □

Exercise 20A: (10) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a complex of R-modules and let H be the homology module $H = Ker(g)/Im(f)$. If F is an exact functor from R-mod to R-mod then the homology of the complex $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is $F(H)$.

Corollary: Let $R \to T$ be a homomorphism of commutative rings which makes T into a flat R-module. If A and B are R-modules then

$$T \otimes_R Tor_n^R(A, B) \simeq Tor_n^T(A \otimes_R T, B \otimes_R T) \ .$$

Proof: Let $C = B \otimes_R T$. Then, by the Proposition, we obtain

$$Tor_n^T(A \otimes_R T, B \otimes_R T) \simeq Tor_n^R(A, B \otimes_R T) \ .$$

So, it's enough to show

$$Tor_n^R(A, B \otimes_R T) \simeq T \otimes_R Tor_n^R(A, B) \ .$$

Now, $T \otimes_R -$ is an exact functor, so, in view of the Exercise above, $T \otimes Tor_n^R(A, B)$ is the n-th homology module of the complex $T \otimes_R (P_\bullet \otimes_R B)$, where $P_\bullet$ is a projective resolution of A. But, that is exactly the homology of the complex $P_\bullet \otimes_R (T \otimes_R B)$, where $P_\bullet$ is a projective resolution of A. But, by definition, this is $Tor_n^R(A, T \otimes_R B)$ which is what we wanted to show. $\square$

Exercise 20B: (Local-Global Principle)

Let M be an R-module. The following are equivalent:

1) (5) $M = 0$;

2) (5) $M_\wp = 0$ for all prime ideals $\wp$ of R;

3) (5) $M_m = 0$ for all maximal ideals m of R.

Prove the "Local - Global Principle" and use it to show the following; If A is an R-module we write $A_\wp = A \otimes R_\wp$. Then, the following are equivalent:

1) (5) $Tor_n^R(A, B) = 0$;

2) (5) $Tor_n^{R_\wp}(A_\wp, B_\wp) = 0$ for all prime ideals $\wp$ of R;

3) (5) $Tor_n^{R_m}(A_m, B_m) = 0$ for all maximal ideals m of R.

The Other Functors!

Let F be a covariant LEFT exact functor from R-mod to R-mod. (Our canonical example of this is the functor $Hom_R(M, -)$). We would like to see what the covariant universal cohomological δ-functor associated to F looks like, if it even exists!

We mimic what we did in the earlier situation: First pick an object $A \in R$-mod and then form a (right) *injective resolution* of A, i.e. a non-negative cochain complex $(I^\bullet, d^\bullet)$ of injective R-modules, i.e.

$$0 \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots$$

and an augmentation map $\epsilon : A \to I^0$ such that

$$0 \to A \xrightarrow{\epsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots$$

is an EXACT complex.

We apply the left exact functor F to this injective resolution $I^\bullet$ of A and we get a complex

$$0 \to F(I^0) \xrightarrow{F(d^0)} F(I^1) \to \cdots$$

and the i-th cohomology of this complex is denoted $(R^i F)(A)$. It is not difficult to see that $(R^0 F)(A) \simeq F(A)$.

Definition: Let $\mathfrak{C}$ be any category. We can consider a new category $\mathfrak{C}^{op}$ (the *opposite category*). The objects of $\mathfrak{C}^{op}$ are the objects of $\mathfrak{C}$, but the morphisms (and their composition) are reversed, i.e. $\mathfrak{C}^{op}(A, B) = \mathfrak{C}(B, A)$.

So, as to not get terribly confused: if $f \in \mathfrak{C}(B, A)$ we write $f^{op} \in \mathfrak{C}^{op}$ as the corresponding morphism. Thus, if $f^{op} \in \mathfrak{C}^{op}(A, B)$, $g^{op} \in \mathfrak{C}^{op}(B, C)$ then $g^{op} \circ f^{op} = (f \circ g)^{op}$.

It's easy to see that: f monic $\Leftrightarrow$ f^{op} epic; f epic $\Leftrightarrow$ f^{op} monic.

Also, "opposites" interchange kernels and cokernels, initial and terminal objects, projective and injective objects.

The point of introducing this idea of the opposite category is the following: If F is a (covariant) left exact functor from the category $\mathfrak{A}$ to the category $\mathfrak{B}$ then there is a functor F^{op} which goes from $\mathfrak{A}^{op}$ to $\mathfrak{B}^{op}$. This functor takes every object to the object that F took it to. On the other hand, if $f^{op} \in \mathfrak{A}^{op}(A, B)$ (i.e. $f \in \mathfrak{A}(B, A)$), then $F(f) \in \mathfrak{B}(F(B), F(A))$, so $F(f)^{op} \in \mathfrak{B}^{op}(F(A), F(B))$, so we define

$$F^{op}(f^{op}) = (F(f))^{op} \ .$$

This new functor F^{op} is easily shown to be a right exact covariant functor. Since injectives become projectives, one can use these new categories, and this new functor, to define $L_i F^{op}$. It's not all that hard to show that $R^i F \simeq (L_i F^{op})^{op}$.

We can apply all of this to the covariant left exact functor $Hom_R(M, -) = F_M$ and prove that its right derived functors are a universal cohomological δ-functor. I won't go through the details.

The right derived functors of $F_M = Hom_R(M, -)$ are denoted, in the non-generic way, by

$$R_i F_M := Ext^i_R(M, -) \text{ where } Ext^0_R(M, -) \simeq Hom_R(M, -) \ .$$

Since we showed earlier (it was, in fact, our definition!) that P is a projective R-module $\Leftrightarrow Hom_R(P,-)$ is an exact functor, and since the higher right derived functors of an exact functor are all 0, we obtain:

Proposition: TFAE

1) P is a projective R-module;
2) $Hom_R(P,-)$ is an exact functor;
3) $Ext^i_R(P,-) = 0$ for all $i \geq 1$;
4) $Ext^1_R(P,-) = 0$.

Proof: 1) $\Leftrightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) are all obvious. So, it remains to prove 4) $\Rightarrow$ 1).

So, let P be a module with the property that $Ext^1(P,-) = 0$ and let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of R-modules. The long exact sequence of cohomology gives

$$0 \to Hom_R(P,A) \to Hom_R(P,B) \to Hom_R(P,C) \to Ext^1_R(P,A) \to \cdots$$

Since $Ext^1_R(P,A) = 0$ we get that

$$0 \to Hom_R(P,A) \to Hom_R(P,B) \to Hom_R(P,C) \to 0$$

is exact, i.e. $Hom_R(P,-)$ is an exact functor. $\square$

It is interesting to note that the Ext functors can also recognize injective modules. Recall that our definition of injectives was: B is an injective R-module $\Leftrightarrow Hom_R(-,B)$ is an exact (contravariant) functor.

At this stage of the game it is hard to see what Ext could say about such functors since it is not defined in terms of them at all! Nevertheless we have the following:

Proposition: TFAE

1) B is an injective R-module;
2) $Hom_R(-,B)$ is an exact functor;

3) $Ext^i_R(A,-)$ vanishes on B for every R-module A and every $i \geq 1$;

4) $Ext^1_R(A,-)$ vanishes on B for every R-module A.

Proof: 1) $\Leftrightarrow$ 2) is the definition.

2) $\Rightarrow$ 3): We want to show that $Ext^i_R(A,B) = 0$ for any R-module A. To find these Ext modules we need an injective resolution of B. But, B is injective, so

$$0 \to B \to 0 \to \cdots$$

is an injective resolution of B. It follows from this that $Ext^i_R(A,B) = 0$ for all $i > 0$ and any A.

3) $\Rightarrow$ 4) is obvious.

4) $\Rightarrow$ 1): We're given that $Ext^1_R(M,B) = 0$ for any R-module M. It's enough to show that, if

$$0 \to A'' \xrightarrow{i} A \to A' \to 0$$

is exact, then

$$0 \to Hom_R(A',B) \to Hom_R(A,B) \xrightarrow{i^*} Hom_R(A'',B) \to 0$$

is exact. Equivalently, i^* is surjective.

Let $0 \to B \to I \to B' \to 0$ be a short exact sequence of R-modules with I injective. From the long exact cohomology sequence, we have, for any R-module M,

$$0 \to Hom_R(M,B) \to Hom_R(M,I) \to Hom_R(M,B') \to Ext^1_R(M,B) = 0$$

if we apply this in turn, with $M = A', A'', A$ we obtain the following large commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Hom_R(A', B) & \to & Hom_R(A', I) & \to & Hom_R(A', B') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Hom_R(A, B) & \to & Hom_R(A, I) & \to & Hom_R(A, B') & \to & 0 \\
 & & {\scriptstyle i^*}\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Hom_R(A'', B) & \to & Hom_R(A'', I) & \to & Hom_R(A'', B') & \to & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

(the middle column arising from the fact that I is injective.) By the Snake Lemma, i^* is surjective, as we wanted to show. □

To get some idea about these functors Ext^i_R, we make some calculations in the case of $R = \mathbb{Z}$.

Proposition: For every abelian group A and B we have

$$Ext^n_{\mathbb{Z}}(A, B) = 0 \text{ for } n \geq 2 \ .$$

Proof: We need an injective resolution of B. We embed B into an injective abelian group, I. Since the quotient of injective $\mathbb{Z}$-modules is again injective, we get an exact sequence

$$0 \to B \to I \to I' \to 0$$

with I and I' both injective.

Thus,

$$0 \to I \to I' \to 0 \to 0 \to \cdots$$

is an injective resolution of B. Applying the functor $Hom_R(A, -)$ to this resolution and taking cohomology gives the result. □

As was the case with the Tor functors, the Ext functors also preserve the multiplication mappings. To be more precise

Lemma: Let M be an R-module and let $\phi_{r,M} : M \to M$ be multiplication by $r \in R$. Then

$$Ext^i_R(\phi_{r,M}) : Ext^i_R(A, M) \to Ext^i_R(A, M)$$

is again multiplication by r.

The proof is similar to that we did for Tor and so will be omitted.

What about the functors $Hom_R(-, M)$. These are contravariant functors and we've not discussed universal anything for them!!! These functors are contravariant and left exact.

However, quite generally, if $F : \mathfrak{A} \to \mathfrak{B}$ is any contravariant left exact functor. This gives rise to a COVARIANT left exact functor from $\mathfrak{A}^{op} \to \mathfrak{B}$ (in the obvious way) and as such, it has right derived functors. I don't want to go into the details. Suffice it to say that one can calculate the cohomology functors associated to the functors $Hom_R(-, M) = G_M$, as follows:

for any R-module A, form a projective resolution of A

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$$

and apply the functor $Hom_R(,M)$ to the resolution $P_\bullet$ to get the complex

$$0 \to Hom_R(P_0, M) \to Hom_R(P_1, M) \to \cdots \ .$$

The cohomology of this complex are the right derived functors of the functors G_M. It is traditional to denoted these functors in the "non-generic" way $Ext^i_R(-, M)$.

Having done that we run into the same problem we had with the Tor's. Fortunately, it doesn't matter in this case either!

In Summary: $Ext^i_R(A, B)$ can be calculated in one of two ways:

take an injective resolution of B and apply the functor $Hom_R(A, -)$ to that resolution and take the ith cohomology;

or, take a projective resolution of A, apply the contravariant functor $Hom_R(-,B)$ to that resolution and take the cohomology of it.

The point is that either way leads to the same module.

Exercise 20C: Calculate the module $Ext^1_{\mathbb{Z}}(A,B)$ when A is a finitely generated abelian group.

Now, localization worked well with the functors Tor and we should check to see what happens with the functors Ext. We need to be a bit more careful in this case.

Lemma: Let R be a noetherian ring and let S be a multiplicatively closed subset of R. Let A be a finitely generated R-module and B any R-module.

There is an isomorphism

$$\phi : S^{-1}Hom_R(A,B) \to Hom_{S^{-1}R}(S^{-1}A, S^{-1}B)$$

defined by: if $f \in Hom_R(A,B)$ and $s \in S$ so that $f/s \in S^{-1}Hom_R(A,B)$, then

$$(\phi(f/s))(a/s_1) = f(a)/ss_1 \ .$$

Proof: There are lots of simple details to check, none of them very difficult, in order to verify that ϕ (as defined) is a homomorphism. That depends not at all on any of the extra hypotheses.

Now to the proof that, with the hypotheses above, ϕ is an isomorphism.

We first consider the case where $A = R$. But, this is obvious since $Hom_R(R,B) \simeq B$ and ϕ is the identity map.

Now, by the additive properties of Hom we get that ϕ is an isomorphism if $A \simeq R^n$, for any n.

Since A is finitely generated, we can find an exact sequence

$$R^m \xrightarrow{g} R^n \xrightarrow{f} A \to 0$$

Consider the two functors, $S^{-1}Hom_R(-,B)$ and $Hom_{S^{-1}R}(S^{-1}-, S^{-1}B)$ and apply each of them to the sequence above. We have, using the notation M_S, for $S^{-1}M$:

$$\begin{array}{ccccccc}
0 & \to & Hom_R(A,B)_S & \xrightarrow{f^*} & Hom_R(R^n,B)_S & \xrightarrow{g^*} & Hom_R(R^m,B)_S \\
 & & & & \simeq\downarrow & & \simeq\downarrow \\
0 & \to & Hom_{R_S}(A_S,B_S) & \to & Hom_{R_S}(R^n_S,B_S) & \xrightarrow{g^*} & Hom_{R_S}(R^m_S,B_S)
\end{array}$$

We are then done by the five lemma.

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Lemma: Let R be a noetherian ring and let S be a multiplicatively closed subset of R. Let A be a finitely generated R-module and B any R-module.

There is an isomorphism

$$\phi : S^{-1}Hom_R(A,B) \to Hom_{S^{-1}R}(S^{-1}A, S^{-1}B)$$

defined by: if $f \in Hom_R(A,B)$ and $s \in S$ so that $f/s \in S^{-1}Hom_R(A,B)$, then

$$(\phi(f/s))(a/s_1) = f(a)/ss_1 \ .$$

Proof: There are lots of simple details to check, none of them very difficult, in order to verify that ϕ (as defined) is a homomorphism. That depends not at all on any of the extra hypotheses.

Now to the proof that, with the hypotheses above, ϕ is an isomorphism.

We first consider the case where $A = R$. But, this is obvious since $Hom_R(R,B) \simeq B$ and ϕ is the identity map.

Now, by the additive properties of Hom we get that ϕ is an isomorphism if $A \simeq R^n$, for any n.

Since A is finitely generated and R is noetherian, we can find an exact sequence

$$R^m \xrightarrow{g} R^n \xrightarrow{f} A \to 0$$

Consider the two functors, $S^{-1}Hom_R(-,B)$ and $Hom_{S^{-1}R}(S^{-1}-, S^{-1}B)$ and apply each of them to the sequence above. We have, using the notation M_S, for $S^{-1}M$:

$$\begin{array}{ccccccc}
0 & \to & Hom_R(A,B)_S & \xrightarrow{f^*} & Hom_R(R^n,B)_S & \xrightarrow{g^*} & Hom_R(R^m,B)_S \\
 & & & & \simeq\downarrow & & \simeq\downarrow \\
0 & \to & Hom_{R_S}(A_S,B_S) & \to & Hom_{R_S}(R^n_S,B_S) & \xrightarrow{g^*} & Hom_{R_S}(R^m_S,B_S)
\end{array}$$

We are then done by the five lemma.

We are now ready to state the proposition which shows how the Ext functors behave under localization.

Proposition: Let R be a noetherian ring and let A be a finitely generated R-module. Suppose that B is any R-module and S is a multiplicatively closed subset of R.

There is an isomorphism:

$$\phi_n : S^{-1}(Ext^n_R(A,B)) \xrightarrow{\simeq} Ext^n_{S^{-1}R}(S^{-1}A, S^{-1}B) \ .$$

Proof: We can find a projective resolution $F_\bullet \to A$ by finitely generated free R-modules. Since $S^{-1}(-) = - \otimes_R S^{-1}R$ is an exact functor ($S^{-1}R$ is a flat R-module), we get that

$$S^{-1}F_\bullet \to S^{-1}A \text{ is a finitely generated free resolution of } S^{-1}A \ .$$

We compute $Ext^n_R(A,B)$ by using the complex $Hom_R(F_\bullet, B)$ and we can calculate $Ext^n_{S^{-1}R}(S^{-1}A, S^{-1}B)$ by using the complex $Hom_{S^{-1}R}(S^{-1}F_\bullet, S^{-1}B)$. By using the Lemma above and the fact that $S^{-1}(-)$ is an exact functor, the proposition follows. □

> *Aside:* Why the name Ext? I don't want to go into this, but the name comes from the fact that $Ext^1_R(A,B)$ describes the set (modulo some equivalence relation) of short exact sequences of R-modules
>
> $$0 \to B \to M \to A \to 0 \ .$$
>
> The same sort of thing is true of $Ext^n_R(A,B)$ but for longer exact sequences.
>
> You can read about this in most books on homological algebra, but Weibel has a good discussion.

Various Dimensions

We can use some of the ideas above to give various notions for the dimension of a commutative ring. This is quite a different approach from the geometric method we introduced last year, i.e. using Krull dimension. It will be interesting to compare the various notions.

Definitions: Let A be an R-module.

1) The *projective dimension* of the module A, denoted $pd(A)$ is the minimum n, if it exists, such that A has a projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0$$

So, e.g. A is a projective R-module $\Leftrightarrow$ $pd(A) = 0$. If no such n exists we say $pd(A) = \infty$.

2) The *injective dimension* of the module A, denoted $id(A)$, is the minimum n, if it exists, such that A has an injective resolution

$$0 \to A \to I^0 \to \cdots \to I^n \to 0 .$$

So, e.g. A is an injective R-module $\Leftrightarrow$ $id(A) = 0$. If no such n exists we say $id(A) = \infty$.

(There is a third notion, *flat dimension*, which can be introduced and discussed, but I won't use it or discuss it in this course.)

Although the notions of projective dimension and injective dimension can differ wildly for any given R-module (just think about $\mathbb{Z}$ as a $\mathbb{Z}$-module) the following theorem shows that the notions are not unconnected!

Theorem: The following numbers are equal for any commutative ring R.

1) $\sup \{pd(B) \mid B \in R-mod\}$;
2) $\sup \{id(B) \mid B \in R-mod\}$;
3) $\sup \{pd(R/I) \mid I \text{ an ideal of } R\}$;
4) $\sup \{d \mid Ext_R^d(A, B) \neq 0 \text{ for some } R\text{-modules } A, B\}$.

This common number (possibly ∞) is called the *global dimension* of the ring A (or the *homological dimension* in some texts).

To prove this theorem we shall need some Lemmas.

Lemma 1: For any R-module A, the following statements are equivalent:

1) $pd(A) \leq d$;
2) $Ext^n_R(A, -) = 0$ for all $n > d$;
3) $Ext^{d+1}_R(A, -) = 0$;
4) If

$$0 \to M_d \to P_{d-1} \to \cdots P_1 \to P_0 \to A \to 0$$

is a resolution of A, with $P_0, \ldots, P_{d-1}$ all projective, then M_d is also projective.

Proof of Lemma: Now 4) $\Rightarrow$ 1) $\Rightarrow$ 2) $\Rightarrow$ 3) are all obvious, so it is enough to show that 3) $\Rightarrow$ 4).

Using the short exact sequence $0 \to K \to P_0 \to A \to 0$ and the fact that P_0 is projective, we get that $Ext^{d+1}(A, B) \simeq Ext^d(K, B)$. Continuing in this fashion, down the resolution of A, we finally get

$$Ext^{d+1}(A, -) \simeq Ext^1(M_d, -)\ .$$

Thus, the hypothesis imply that $Ext^1(M_d, -) = 0$. But, by an earlier Lemma, this vanishing characterizes projective R-modules. So, M_d is a projective R-module, as we wanted to show. □

In exactly the same way we have

Lemma 1′ : For any R-module B, the following are equivalent:
1) $id(B) \leq d$;
2) $Ext^n_R(-, B) = 0$ for all $n > d$;
3) $Ext^{d+1}_R(-, B) = 0$;
4) If

$$0 \to B \to I^0 \to \cdots \to I^{d-1} \to L^d \to 0$$

is a resolution of B with $I^0, \ldots, I^{d-1}$ all injective, then L^d is also injective.

Exercise 21A: Show that if m is not a square-free integer then the ring $R = \mathbb{Z}/m$ has infinite global dimension.

Lemma 2: The R-module B is injective $\Leftrightarrow$ $Ext^1_R(-,B)$ vanishes on all R-modules of the form R/I, I an ideal of R.

Proof: Recall that we showed earlier that B is an injective R-module $\Leftrightarrow$ $Ext^1_R(-,B)$ vanishes on ALL R-modules. So, this is a significant weakening of that earlier result.

Now

$$0 \to I \to R \to R/I \to 0$$

is a short exact sequence, so by the long exact sequence of cohomology we have

$$0 \to Hom(R/I,B) \to Hom(R,B) \to Hom(I,B) \to Ext^1(R/I,B) \to Ext^1(R,B)\ .$$

Since R is free, $Ext^1(R,B) = 0$.

Now $Ext^1(R/I,B) = 0$ implies that $Hom(R,B) \to Hom(I,B) \to 0$ is exact. But, by Exercise 4d, if this is true for every ideal I of R then B is an injective R-module. Thus, we are done with the lemma.

We now return to the proof of the theorem.

Proof: If, for some n, $Ext^{n+1}(A,B) \neq 0$ then by the Lemmas 1 and $1'$ we have $pd(A) > n$ and $id(B) > n$. I.e. the two lemmas show that $sup(2) = sup(4) = sup(1)$. Consequently, $sup(2) \geq sup(3)$.

Suppose that $sup(2) > sup(3)$. Then, we may as well suppose that $sup(3)$ is finite, say $= d$. I.e.

$$d = sup\{pd(R/I) \mid I \text{ is an ideal of } R\}\ .$$

Since we are assuming that $sup(2) > d$, we may suppose there is an R-module B such that $id(B) > d$.

Consider a resolution of this B. We have:

$$0 \to B \to I^0 \to I^1 \to \cdots \to I^{d-1} \to M \to 0$$

where we choose $I^0, \ldots, I^{d-1}$ injective. By the hypothesis, M cannot be injective.

However, for all ideals I of R we have

$$0 = Ext^{d+1}(R/I,B) \simeq Ext^1(R/I,M)\ .$$

But, by the Lemma 2 above, $Ext^1(R/I, M) = 0$ for all ideals I of R implies that M is injective. This contradiction establishes the result.

Exercise 21B: Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of R-modules. Then

1)

$$pd(B) \leq max\{pd(A), pd(C)\}$$

with equality EXCEPT when $pd(C) = pd(A) + 1$.

2)

$$id(B) \leq max\{id(A), id(C)\}$$

with equality EXCEPT when $id(A) = id(C) + 1$.

3) If $A = \oplus A_i$ then $pd(A) = sup\{pd(A_i)\}$.

4) If $gl.dim.(R) = \infty$, prove that there is an R-module M with $pd(M) = \infty$.

Remarks: It is easy to see that a field has global dimension 0 and that the same is true for a finite direct sum of fields. It is not trivial to prove that the converse is also true. I.e. a commutative ring has global dimension 0 if and only if it is a finite direct sum of fields. (The proof in one direction uses the Wedderburn Structure theorem for rings with DCC, and I won't go into that at this point.)

It is also easy to see that the ring $\mathbb{Z}$ has global dimension 1 since there are modules which are not injective (i.e. in this case, divisible) and every module has an injective resolution of the type

$$0 \to A \to I^0 \to I^1 \to 0$$

since the quotient of a divisible module is always divisible (and hence, over $\mathbb{Z}$, is injective). We only used the fact that $\mathbb{Z}$ was a pid to prove our results about injectivity and divisibility, so it is also true that $k[X]$ has global dimension 1 if k is a field. We have already observed that $\mathbb{Z}/m$ has infinite global dimension if m is not square free.

sect11b.tex

In order to get more information on dimension we'll use several "change of ring" theorems, which compare projective dimensions of modules over related rings.

Theorem: (General Change of Rings Theorem)

Let $\phi : R \to S$ be a homomorphism of commutative rings and let M be an S-module. Then

$$pd_R(M) \leq pd_R(S) + pd_S(M) \ .$$

(a sort of "triangle inequality".)

Proof: If $pd_R(S)$ or $pd_S(M)$ is infinite, there is nothing to prove, so we may as well assume that both are finite. Let $pd_R(S) = n$ (which will remain fixed for the discussion) and let $pd_S(M) = k$. We do induction on k.

k = 0: Then M is a projective S-module. So, there is another projective S-module M' such that $M \oplus M'$ is a free S-module.

Since $pd_R(M) \leq pd_R(M \oplus M') = pd_R(S) = n = n + 0$, we are done in this case and the induction is started.

So, assume the theorem is true for all S-modules M for which $pd_S(M) < k$, and now suppose that $pd_S(M) = k > 0$.

We have

$$0 \to K \to F \to M \to 0$$

an exact sequence, where F is a free S-module. Thus, $pd_S(F) < pd_S(M)$ and so, by the exercise above, $pd_S(M) = pd_S(K) + 1$. Hence, by induction applied to the S-module K, we have

$$pd_R(K) \leq n + (k-1) \ .$$

Now, since the sequence above is exact, is is also exact as a sequence of R-modules. Note also that since F is a free S-module, $pd_R(F) = pd_R(S) = n$.

There are three cases to consider:

a) $pd_R(K) > pd_R(F) = n$:

Then, again by the Exercise, we don't have equality in the inequality! so,

$$pd_R(M) = pd_R(K) + 1 \leq n + (k-1) + 1 \leq n + k$$

and we are done in this case.

b) $pd_R(K) = pd_R(F) = n$:

Then, either $pd_R(M) = pd_R(F) = n$, and we are done or $pd_R(M) > pd_R(F)$. But, in this latter case the inequality of the exercise above is not an equality and we get $pd_R(M) = pd_R(K) + 1 = n + 1$. Since this is $\leq n + k$ for $k \geq 1$ we are done in this case also.

c) $pd_R(K) < pd_R(F) = n$:

So, either $pd_R(F) = pd_R(M) = n$ and we are done, or we don't have equality in the inequality and

$$pd_R(M) = pd_R(K) + 1 < n + 1 \leq n$$

and we are done in this case also.

That completes the proof of the General Change of Rings Theorem. □

Remark: Note that in case $S = R[x]$ and ϕ is the natural inclusion of R into $R[x]$ we get nice results because S is a FREE R-module and so $pd_R(S) = 0$.

It follows that if M is an $R[x]$-module and we consider it also as an R-module on which x acts trivially, then

$$pd_R(M) \leq pd_{R[x]}(M) \ .$$

As a simple corollary to this we get:

Corollary: If R is any commutative ring then

$$\text{global dimension of } R \leq \text{ global dimension of } R[x] \ .$$

Proof: Let N be any R-module. Then N may be viewed as an $R[x]$-module on which x acts trivially. If we restrict the scalars on this $R[x]$-module to R we obviously get back N with its original R-module structure.

Since N was an arbitrary R-module and $pd_R(N) \leq pd_{R[x]}(N)$, that proves the result. □

Another interesting change of rings result comes in a case when $pd_R(S)$ is small.

Proposition: Let $x \in R$ and suppose that x is neither a unit nor a zero-divisor in R.

The $pd_R(R/(x)) = 1$.

Proof: We have an exact sequence of R-modules

$$0 \to (x) \to R \to R/(x) \to 0 \qquad (*)$$

and since x is not a zero-divisor $\phi : R \to (x)$ by $\phi(1) = x$ is an isomorphism of R-modules. Since x is not a unit, $R/(x) \neq 0$ and so $(*)$ is a free (hence projective) resolution of $R/(x)$. Thus, $pd_R(R/(x)) \leq 1$.

However, $R/(x)$ is not a projective R-module since x annihilates it and a projective R-module has no annihilator besides 0. □

So, if we employ our general change of rings theorem and we let N be an $S = R/(x)$-module (i.e. an R-module which is annihilated by x) then $pd_R N \leq 1 + pd_S(N)$. But, it turns out that we can do much better.

Theorem: (First Change of Rings Theorem)

Let $S = R/(x)$, where x is neither a unit nor a zero-divisor in the ring R and let A be any S-module with **finite** projective dimension.

Then

$$pd_R A = pd_S A + 1 \ .$$

Proof: From what we have already observed, it suffices to prove that

$$pd_R A \geq pd_S A + 1 \ .$$

We do induction on $n = pd_S A$.

n=0 : We only need to show that A has positive projective dimension over R. But this is clear since A is an R-module which is annihilated by x, so cannot be projective as an R-module.

n=1 : Since $pd_S A = 1$ we want to show that $pd_R A \geq 2$.

So, suppose not, i.e. suppose that $pd_R A \leq 1$. Then we have an exact sequence

$$0 \to K \to F \xrightarrow{f} A \to 0 \qquad (\dagger)$$

where F is free. Since we are assuming that $pd_R A \leq 1$ we must also have that K is projective.

Using the long exact sequence of homology associated to the functor $- \otimes_R R/(x)$ applied to the sequence (†) we get

$$Tor_1(F, A) \to Tor_1(A, R/(x)) \to K \otimes_R R/(x) \to F \otimes_R R/(x) \to A \otimes_R R/(x) \to 0 \ .$$

Since F is projective, $Tor_1(F, A) = 0$ and by earlier results, $M \otimes_R (R/I) \simeq M/IM$. But notice that $A/xA \simeq A$ since x acts trivially on A. So, rewriting the sequence above we get:

$$0 \to Tor_1^R(A, R/(x)) \to K/xK \to F/xF \to A \to 0$$

which is an exact sequence of S-modules.

Now we can compute $Tor_1^R(A, R/(x))$ from the resolution

$$0 \to (x) \xrightarrow{i} R \to R/(x) \to 0 \ .$$

It is the 1st homology module of the complex

$$\begin{array}{ccccccc} 0 & \to & (x) \otimes_R A & \xrightarrow{i \otimes id} & R \otimes_R A & \to & 0 \\ & & \downarrow\simeq & & \downarrow\simeq & & \\ 0 & \to & A & \xrightarrow{x} & A & \to & 0 \end{array}$$

where "x " is multiplication by x. But we observed that x annihilates A, so

$$Tor_1^R(A, R/(x)) \simeq A \ .$$

Thus, the sequence above becomes

$$0 \to A \to K/xK \to F/xF \to A \to 0 \ .$$

Since K/xK and F/xF are projective S-modules and $pd_S A = 1$ we obtain that the last "kernel" is also projective, i.e. A is a projective S=module. That contradicts our assumption that $pd_S A = 1$. That contradiction establishes the result in this case.

general n : So, suppose that $pd_S A = n \geq 2$. Let F be a free S-module mapping onto A and consider

$$0 \to K \to F \to A \to 0 \qquad (\dagger\dagger)$$

Now $0 = pd_S F < pd_S A$ thus (by the exercise assigned last time) we have $n = pd_S A = pd_S K + 1$. We can apply induction to the module K and get $pd_R K = 1 + pd_S K$.

Hence, considering $(\dagger\dagger)$ as an exact sequence of R-modules we have

$$1 = pd_R F \leq max\{pd_R K, pd_R A\} = max\{1 + pd_S K, pd_R A\} = max\{1 + n - 1, pd_R A\} \ .$$

Since $n \geq 2$, we obtain an inequality above, and so, by our earlier exercise again,

$$pd_R A = 1 + pd_R K = 1 + n$$

as we wanted to show. □

Remark: Notice that the theorem is definitely false if A does not have finite projective dimension.

To see this, consider $4 \in \mathbb{Z}$, then 4 is neither a unit nor a zero-divisor in $\mathbb{Z}$. Thus $\mathbb{Z} \to \mathbb{Z}/(4)$ is an extension of the type above.

If we let M be the $\mathbb{Z}$-module $M = \mathbb{Z}/(2)$, then by an earlier calculation we have $pd_{\mathbb{Z}/(4)} M = \infty$. However, $pd_{\mathbb{Z}} M = 1$.

Theorem: (Second Change of Rings Theorem)

Let $S = R/(x)$ with x neither a unit nor a zero-divisor in R. Let A be any R-module and suppose x is not a zero divisor on A. Then

$$pd_R(A) \geq pd_S(A/xA) \ .$$

Proof: If $pd_R A = \infty$ there is nothing to prove.

So, suppose that $pd_R A = n$ and we proceed by induction on n. The case $n = 0$ is trivial since A projective implies that $A \otimes R/(x) \simeq A/xA$ is projective as an S-module. So, we even get equality in this case.

Now suppose that $pd_R(A) = n > 0$ and let F be a free R-module. We have

$$0 \to K \to F \to A \to 0$$

and $pd_R K = pd_R A - 1 = n - 1$.

Now, as we saw in the earlier theorem, from the long exact sequence of homology associated to the functor $- \otimes_R R/(x)$ we get

$$0 \to Tor_1^R(A, R/(x)) \to K/xK \to F/xF \to A/xA \to 0$$

and

$$Tor_1^R(A, R/(x)) = \{a \in A \mid xa = 0\} \ .$$

But, by assumption, this $Tor_1 = 0$. Thus

$$0 \to K/xK \to F/xF \to A/xA \to 0$$

is exact. By induction $pd_R K = n - 1 > pd_S K/xK$. Since F/xF is a free S-module we conclude that $pd_S A/xA \leq n$ (just piece the resolution of K/xK onto this one!) and so we are done. □

sect12a.tex

We can use these two Change of Rings Theorems to get a sharper result that we had earlier about the ring extension $R \to R[x]$. Recall that we write $A[x]$ for the module $A \otimes_R R[x]$.

Proposition: Let A be R-module, then

$$pd_R A = pd_{R[x]} A[x] \ .$$

Proof: Consider $\phi : R[x] \to R$ by $\phi(x) = 0$. (i.e. not the inclusion!) and the $R[x]$-module $A[x]$.

It is easy to see that x is **not** a zero-divisor on the module $A[x]$ (multiplication by x just "shifts" things to the right!). Since $A[x]/(x) \simeq A$ we can apply the Second Change of Rings Theorem to get

$$pd_{R[x]} A[x] \geq pd_R A \ .$$

As for the other inequality, let $P_\bullet \to A \to 0$ be a projective resolution of A as an R-module. Since $R[x]$ is a free (hence flat) R-module we get that $P_\bullet[x]$ is a projective resolution of $A[x]$. Thus,

$$pd_R A \geq pd_{R[x]} A[x]$$

as we wanted to show. $\square$

This is the opening salvo in the proof of the famous

Theorem: If R is a commutative ring then

$$\text{global dimension of } R[x_1, \ldots, x_n] = n + \text{ global dimension of } R \ .$$

In particular, if k is a field then (Hilbert's Syzygy Theorem)

$$gl.dim.k[x_1, \ldots, x_n] = n \ .$$

We'll see later that a **graded** projective module is free and this was Hilbert's interest in the question. Explain!

Proof: It's enough to consider the case of $T = R[x]$.

If $gl.dim.R = \infty$ then $gl.dim.R[x] = \infty$ as well since $pd_R A = pd_R A[x]$ for any R-module A.

So, we may as well assume that $gl.dim.R = n < \infty$. Since x is not a zero divisor in T the change of rings $R[x] \to R$ is as given in the First Change of Rings Theorem. Thus, if A is an R-module with $pd_R A < \infty$ we can apply the First Change of Rings theorem to obtain

$$pd_{R[x]} A = 1 + pd_R A \ .$$

Since A was an arbitrary R-module we obtain

$$gl.dim.R[x] \geq 1 + gl.dim.R \ .$$

To show the other inequality, let M be any $T = R[x]$-module. Let $U(M) = M$, but just with the induced R-module structure. Then, $R[x] \otimes_R U(M)$ is an $R[x]$-module by action on the left factor.

There is an R-linear map

$$\beta : R[x] \otimes_R U(M) \to R[x] \otimes_R U(M)$$

coming from the R-bilinear map

$$R[x] \times U(M) \longrightarrow R[x] \otimes_R U(M)$$

$$(t, m) \longrightarrow t[x \otimes m - 1 \otimes xm] \ .$$

Now, the $R[x]$-module structure on $R[x] \otimes_R U(M)$ is, as we mentioned above,

$$t_1(t_2 \otimes m) = (t_1 t_2) \otimes m \ .$$

Noting this, it is an easy verification that β is also an $R[x] = T$-module homomorphism as well.

The multiplication map

$$\mu : R[x] \otimes_R U(M) \longrightarrow M$$

given by, $\mu(t \otimes m) = tm$, is also clearly an $R[x] = T$-module homomorphism as well.

Claim: For any $R[x]$-module M let $N = R[x] \otimes_R U(M)$. The following sequence is exact:

$$0 \to N \xrightarrow{\beta} N \xrightarrow{\mu} M \to 0 \ .$$

Suppose the Claim is verified! Then by an earlier Corollary, (since $N = U(M)[x]$)

$$pd_{R[x]} N = pd_R U(M) \leq n$$

So, from the long exact cohomology sequence applied to the exact sequence in the claim, we obtain

$$\cdots \to Ext^d_{R[x]}(N, A) \to Ext^{d+1}_{R[x]}(M, A) \to Ext^{d+1}_{R[x]}(N, A) \to \cdots$$

and $Ext^d_{R[x]}(N, A) = 0$ for all $d \geq n + 1$. Thus, $Ext^{d+1}_{R[x]}(M, A) = 0$ for all $d + 1 \geq n + 2$. Thus, $pd_{R[x]} M \leq n + 1$ and so

$$gl.dim.R[x] \leq gl.dim.R + 1$$

as we wanted to show.

Thus it only remains to prove the Claim.

Pf. (of claim) Since $\beta(1 \otimes m) = m$, the map μ is definitely onto. Also, $\mu\beta(t \otimes m) = \mu(t(x \otimes m - 1 \otimes xm)) = t(xm - xm) = 0$.

So, it's enough to show that β is 1-1 and $Ker(\mu) \subseteq Im(\beta)$.

To do that, notice first that since $R[x]$ is a free R-module with basis $\{1, x, x^2, \ldots\}$, every element $f \in N$ can be uniquely written

$$f = x^n \otimes m_n + x^{n-1} \otimes m_{n-1} + \cdots + x \otimes m_1 + 1 \otimes m_0$$

with $m_n \neq 0$ and $m_i \in U(M)$.

Since $\beta(x^n \otimes m_n) = x^{n+1} \otimes m_n - x^n \otimes m_n'$, it follows that the "leading term" of $\beta(f)$ is $x^{n+1} \otimes m_n$ (with $m_n \neq 0$). Thus, $\beta(f) = 0$ iff $f = 0$, i.e. β is $1 - 1$.

We now prove, by induction on $n = deg f$ that if $n \in Ker(\mu)$ then $n \in Im(\beta)$.

Since $\mu(1 \otimes m) = m$, nothing of degree 0, except 0, is in $Ker(\mu)$, and we certainly have that 0 is in $im(\beta)$.

Notice that $\beta(x^{n-1} \otimes m_n) = x^n \otimes m_n - x^{n-1} \otimes m_{n-1}'$, so

$$\mu(f - \beta(x^{n-1} \otimes m_n)) = \mu(f) - \mu\beta(x^{n-1} \otimes m_n) = \mu(f) \ .$$

We see, also, that $f - \beta(x^{n-1} \otimes m_n)$ has degree $n - 1$. Thus, if $\mu(f) = 0$ then $\mu(f - \beta(x^{n-1} \otimes m_n)) = 0$ also. And, by induction, $f - \beta(x^{n-1} \otimes m_n) = \beta(h)$ for some h. But, this implies that $f \in Im(\beta)$, as we wanted to show. □

Our next change of rings theorem will be stated for local rings - since it requires a use of Nakayama's Lemma.

Recall: If (R, m) is a local ring and M is a finitely generated R-module for which $IM = M$ for some ideal I in R, then $M = 0$.
It is not entirely accurate to say that local rings are needed for the use of this theorem. It is for local rings that the statement is true for ANY

ideal I of the ring. There are more occasions in which a result like this is true, but for a limited class of ideals. The most interesting of these is for positively graded rings $R = \oplus_{i=0}^{\infty} R_i$ with $R_0 = k$ a field and $m = \oplus_{i=1}^{\infty} R_i$. In this case if M is a finitely generated graded module and I is a homogeneous ideal contained in m then the same conclusion holds. It is useful to keep this one in mind for the remainder of the theorems in the course.

Note: It is an immediate consequence of this statement of the Nakayama Lemma that if M is a finitely generated R-module and $N \subseteq M$ and $N + IM = M$ then $N = M$. (One simply sets $K = M/N$ and notes $IK = K$. I'll leave the details to the reader.)

Proposition: Let (R, m) be a local ring and let P be a finitely generated projective R-module, then P is a free R-module.

Proof: Notice that P/mP is a finite dimensional $R/m = k$ vector space, say of dimension n. So we can find $x_1, \ldots, x_n \in P$ which form a basis for P/mP. Let $N =< x_1, \ldots, x_n >\subseteq P$. Then $N + mP = P$ and so by Nakayama's Lemma, $N = P$. It remains to show that the x_i are independent.

Now, the map $\phi : R^n \to P$ by $\phi(a_1, \ldots, a_n) = \sum a_i x_i$ is an onto homomorphism. Thus, since P is projective, the map splits and $R^n \simeq P \oplus Ker\phi$. Since $- \otimes_R (R/m)$ is an additive functor, we have

$$k^n \simeq R^n/mR^n \simeq P/mP \oplus Ker(\phi)/mKer(\phi) \simeq k^n \oplus Ker(\phi)/mKer(\phi) \ .$$

Thus, $Ker(\phi)/mKer(\phi) = 0$, i.e. $Ker(\phi) \subseteq mR^n$. Since $R^n \simeq P \oplus Ker(\phi)$ we get:

$$R^n = P + Ker(\phi) = P + mR^n \ .$$

So, again by Nakayama's Lemma, $R^n = P$, as we wanted to show. □

We now come to the Third (and last, for the time being) Change of Rings Theorem. It substantially improves the Second, but with a price in more stringent hypotheses.

Theorem: (Third Change of Rings Theorem)

Let (R, m) be a noetherian local ring and let A be a finitely generated R-module. Let $x \in R$ be neither a unit nor a zero divisor in R and suppose also that x is not a zero divisor on the module A. Then

$$pd_R A = pd_{R/xR}(A/xA) \ .$$

Proof: By the Second Change of Rings theorem we have the inequality "$\geq$". So, it only remains to prove "$\leq$" . To do that we shall work by induction on $n = pd_{R/xR}(A/xA)$.

Case 1: $(n = 0)$ I.e. we are assuming that A/xA is a projective $R/(x)$-module. Since $R/(x)$ is again local, by the proposition above, A/xA is free. We need to show that A is a projective (hence free) R-module. So, let $\overline{u_1}, \ldots, \overline{u_m}$ be an $R/(x)$ basis of A/xA.

Claim: $u_1, \ldots, u_m$ are a basis for A as an R-module.

Pf: Clearly $< u_1, \ldots, u_n > +xA = A$ and so, by Nakayama's Lemma, $< u_1, \ldots, u_n >= A$. We have to show that $u_1, \ldots, u_m$ are independent.

So, suppose that $\sum_{i=1}^{m} r_i u_i = 0$ where $r_i \in R$. By passing to A/xA we conclude that $\overline{r_i} = 0$ in R/xR, i.e. $r_i \in xR$ for all i. So, we can write $r_i = xr_i'$ and we have $0 = \sum r_i u_i = x(\sum r_i' u_i)$. Since x is not a zero divisor on A we get $\sum r_i' u_i = 0$ also.

We may continue this process and in this way we obtain, for each i,

$$(r_i) \subseteq (r_i') \subseteq (r_i'') \subseteq \cdots \ .$$

But, R is noetherian and so these chains all eventually become constant.

Thus, with no loss of generality, we may assume we have (for all r_i and r_i') that $(r_i) = (r_i')$ i.e. $r_i = xr_i'$ and $r_i' = yr_i$ for x as above and $y \in R$. But then $r_i = xr_i' = x(yr_i)$, i.e. $(1 - xy)r_i = 0$. But, $x \in m$ implies that $1 - xy$ is a unit of R. Thus $r_i = 0$ and that finishes the proof for this case of the theorem.

Case 2:$(n > 0)$ We suppose that A is such that A/xA has projective dimension (over R/xR) n and we want to show that the same is true for A as an R-module.

Let F be a free R-module mapping onto A and let K be the kernel, then

$$0 \to K \to F \to A \to 0 \qquad (*)$$

is a short exact sequence.

Now we showed earlier that

$$Tor_1^R(A, R/(x)) = \{a \in A \mid xa = 0\} .$$

But, by assumption, x is not a zero divisor on A so, $Tor_1^R(A, R/(x)) = 0$. So, if we apply the functor $- \otimes_R (R/(x))$ to $(*)$ above we get

$$0 \to K/xK \to F/xF \to A/xA \to 0 .$$

Since $pd_{R/(x)} A/xA = n$ and F/xF is a free $R/(x)$-module we obtain

$$pd_{R/(x)} K/xK = pd_{R/(x)} A/xA - 1 = n - 1 .$$

By induction, $pd_R K = n - 1$. Since A is not free we have, from $(*)$ that $pd_R K + 1 = pd_R A$, i.e. $pd_R A = (n-1) + 1 = n$, as we wanted to show. □

sect12b.tex

Corollary: Let (R, m) be a noetherian local ring and let A be a finitely generated R-module. Suppose $x \in m$ is neither a unit nor a zero divisor in R and not a zero divisor on A. If $pd_R A < \infty$ then

$$pd_R A + 1 = pd_R(A/xA) .$$

Proof: By the Third Change of Rings Theorem, $pd_R A = pd_{R/xR}(A/xA)$ so the latter projective dimension is also finite. So, we can apply the First Change of Rings Theorem to assert that

$$pd_R(A/xA) = 1 + pd_{R/xR}(A/xA) .$$

Another application of the Third Change of Rings Theorem finishes off the proof of the corollary. □

Applications

I would like to give some applications of the ideas of homological algebra to some ideas from commutative algebra. This section will depend on some of the material we covered in the course last year (principal ideal theorem e.g.). From now on R will always denote a commutative noetherian ring and the modules I consider will be finitely generated.

Definition: The elements $\{x_1, \ldots, x_n\} \in R$ are called a *regular sequence on the R-module M* if:

1) $(x_1, \ldots, x_n)M \neq M$;

2) x_1 is not a zero-divisor on M and x_i is not a zero divisor on $M/(x_1, \ldots, x_{i-1})M$ for each $i = 2, \ldots, n-1$.

The length of the longest regular sequence on the module M is called the *grade of M* and denoted $G(M)$. It's not hard to show that if (R, m) is a local ring then $G(R) \leq$ Krull dimension R ($ht(x_1) = 1$ since x_1 is not a zero divisor on R and $ht(x_1, x_2) = 2$ etc. - use the Principal Ideal Theorem.)

Definition: The local ring (R, m) is called a *Cohen-Macaulay* ring if $G(R) =$ Krull dimension R.

If I is an ideal in the noetherian local ring (R, m) then it is easy to see that the minimal number of generators of the ideal I is the dimension of the vector space I/mI. We denote this number by $\nu(I)$. In particular

$$dim_{R/m}(m/m^2) = \nu(m) := \text{ embedding dimension of } R\ .$$

It is a simple application of the Principal Ideal Theorem to see that:

Proposition: If (R, m) is a noetherian local ring then

$$\text{Krull dimension } R \leq \nu(m) = \text{ embedding dimension } R\ .$$

Clearly a ring for which those two numbers are equal deserves special notice.

Definition: A local ring (R, m) is called a *regular local ring* if

$$\text{Krull dimension } R = \text{ embedding dimension } R\ .$$

i.e. $\nu(m) = ht(m)$.

We have a small but important observation about regular local rings.

Theorem: A regular local ring (R, m) is an integral domain.

Proof: If the Krull dimension of R is 0 then $m = 0$ and so R is a field and we're done in this case.

So, suppose that the Krull dimension of R is > 0. Choose $x \in m/m^2$. Then x can be extended to a minimal generating set of m since $\overline{x}$ is not 0 in m/m^2.

Consider $(R/(x), m/(x))$. Clearly, $\nu(m/(x)) = \nu(m) - 1$, so we have that

$$\text{Krull dimension } R/(x) < \text{ Krull dimension } R\ .$$

In fact, by a theorem we proved last year, Krull dimension $R/(x) =$ Krull dimension $R - 1$. Thus, $R/(x)$ is again a regular local ring, this time of Krull dimension $n - 1$.

By induction, it is a domain, i.e. (x) is a prime ideal.

So, if R is **not** a domain, $(x) = \wp$ is a minimal prime ideal for every $x \in m - m^2$.

But, R is noetherian so there are only a finite number of minimal prime ideals in R, call them $\wp_1, \ldots, \wp_t$, and we are saying that

$$m - m^2 \subseteq \wp_1 \cup \ldots \cup \wp_t \text{ i.e. } m \subseteq m^2 \cup \wp_1 \cup \ldots \cup \wp_t\ .$$

But, $m \not\subseteq m^2$ so $m \subseteq \wp_i$ for some i. But, this implies that $d = 0$, which is a contradiction. $\Box$

Corollary: If (R, m) is a regular local ring of dimension d then R is C.M. and any set of minimal generators for m is a regular sequence.

Proof: Let $x_1, \ldots, x_d$ be any (minimal) set of generators for m. Then $x_1 \neq 0$ implies that it is not a zero divisor since R is a domain.

Now $R/(x_1)$ is again a regular local ring (as we saw above) and so is again a domain and x_2 does not become 0 in this ring since the x_i are minimal generators for m. So, $\overline{x_2}$ is not a zero divisor in this ring. We continue the process. $\Box$

The connection with the homological machine makes its entry with the following corollary.

Corollary: If (R, m) is a regular local ring of Krull dimension $= n$ then the global dimension of R is n.

Proof: We do induction on $n =$ Krull dimension of R.

If $n = 0$ then R is a field and so we are done.

We now proceed to the general case. We suppose that the Krull dimension of R is n and that $(x_1, \ldots, x_n) = m$. We know that $R/(x_1)$ is a regular local ring of dimension $n - 1$.

If I is any ideal of R then, since x_1 is not a zero divisor in R it is certainly not a zero divisor on I. Thus, by the Third Change of Rings Theorem,

$$pd_R(I) = pd_{R/(x_1)}(I/x_1 I) \ .$$

Thus, $pd_R(I) \leq n - 1 =$ global dimension $R/(x_1)$.

From the usual short exact sequence of R-modules

$$0 \to I \to R \to R/I \to 0$$

we see that $pd_R(R/I) \leq (n-1) + 1 = n$.

Since, as we saw, the global dimension is also the supremum of the projective dimensions of R-modules of the form R/I, we have that the global dimension of R is finite and $\leq n$.

On the other hand, if M is any $R/(x_1)$-module having projective dimension $n - 1$ then, by the First Change of Rings Theorem, we have

$$pd_R M = 1 + pd_{R/(x_1)} M = 1 + (n-1) = n$$

and that proves the result. □

Regular local rings have played a major role in algebraic geometry. They are the local rings of the simple points on algebraic varieties. Algebraically they have been of interest since introduced by Krull, in the 1930's. Krull proved many interesting things about such rings, but one property eluded him and was a conjecture for 25 years. It states:

Krull's Conjecture: If R is a regular local ring and $\wp$ is a prime ideal of R then $R_\wp$ is again a regular local ring.

The positive solution to Krull's conjecture by Maurice Auslander and David Buchsbaum (and independently J.P. Serre) in the mid 1950's was the first major triumph of the homological entry into commutative algebra and algebraic geometry. I want to give that proof now.

First a simple remark.

Proposition: If R is any commutative ring and S is a multiplicatively closed subset of R, then

$$\text{global dimension } R \geq \text{ global dimension } S^{-1}R\ .$$

Proof: The result follows from the following simple remark: if M is any $S^{-1}R$-module then $M = S^{-1}N$ for some R-module N. (in fact, take $M = N$!)

So, if we take a projective resolution $P_\bullet$ of M as an R-module then $P_\bullet \otimes_R S^{-1}R$ gives a projective resolution of M as an $S^{-1}R$-module. Thus, $pd_{S^{-1}R}M \leq pd_R M$ and the result is immediate from that. □

From this proposition we see what is needed to prove Krull's Conjecture. We start with a regular local ring. We showed above that it has finite global dimension. From the proposition we see that any localization of the regular local ring at a multiplicatively closed set, is again of finite global dimension. In particular if we localize the original regular local ring at a prime ideal we obtain a local ring of finite global dimension. If we knew that the regular local rings were **characterized** by the fact that they had finite global dimension, we would be done.

So, that is what is left to prove – a local ring of finite global dimension is regular.

That result will follow from a string of lemmas. I will also need one remark from the standard theory of associated primes in commutative rings. Let me state it now:

> if R is noetherian and M is finitely generated then the set of zero divisors in R on M is a finite union of prime ideals, each of which is the annihilator of a non-zero element of M.

Lemma 1: Let (R, m) be a local ring and let M be a finitely generated R-module. Let $m_1, \ldots, m_n$ be a minimal set of generators for M and let F be a free R-module on n

generators $\{e_1, \ldots, e_n\}$ mapping onto M by

$$\phi : F \to M \text{ where } \phi(e_i) = m_i \ .$$

Then $Ker(\phi) \subseteq mF = m \oplus \cdots \oplus m$.

Proof: Suppose not, i.e. suppose that $(c_1, \ldots, c_m) \in Ker(\phi)$ and c_1 (say) is not in m.

Then $\sum_{i=1}^{n} c_i m_i = 0$ and so, since c_1 is a unit in R, we have $m_1 = \sum_{i=2}^{n}(c_i/c_1)m_i$, contradicting the minimality of the generating set for m.

Lemma 2: Let (R, m) be a local ring and suppose that m consists entirely of zero-divisors.

If M is any finitely generated R-module then either $pd_R M = 0$ or $pd_R M = \infty$.

Proof: Notice first that if there is a finitely generated module of projective dimension > 1 then there is also a finitely generated module of projective dimension $= 1$ (just take the appropriate syzygy module of the module of finite projective dimension). So, if the theorem is not true, there is a finitely generated module M, with $pd_R M = 1$. Let $\nu(M) = n$ and let F be a free module on n generators mapping onto M. We have:

$$0 \to K \to F \to M \to 0$$

is exact.

Since $pd_R M = 1$ we have that K is a projective R-module, hence free.

Since m consists entirely of zero divisors of R we conclude that $m = ann(z)$ for some $z \in R$.

Now, by Lemma 1, we have that $K \subseteq m \oplus \cdots \oplus m$. But, since $m = ann(z)$ we must have that $zK = 0$. But, this is impossible for a projective R-module.

That contradiction implies that $pd_R M$ is either 0 or ∞. □

sect13a.tex

We are now ready to prove the main result.

Theorem: (Auslander-Buchsbaum, Serre)
Let (R, m) be a noetherian local ring and suppose that $pd_R m < \infty$.

Then (R, m) is a regular local ring and

$$\text{Krull dimension } R = pd_R m + 1 \ .$$

Proof: We shall prove this by double induction on $n = pd_R m$ and $k =$ the Krull dimension of R.

n = 0: In this case, m is a projective (hence free) R-module. But, m is an ideal so this means that $m = (x)$ where x is neither a unit nor a zero divisor in R. Then, by the Principal Ideal Theorem, we have that $ht(m) = 1$ and (by definition) (R, m) is a regular local ring of dimension 1.

k = 0: Let m be the maximal ideal of R. If $m \neq (0)$ then, since the Krull dimension is 0, m consists entirely of zero divisors. But, by hypothesis $pd_R m < \infty$, so by the Lemma 2 above, we must have that $pd_R m = 0$. I.e. m is a free R-module. But, that is impossible if m consists entirely of zero divisors. So, it must be that $m = (0)$, i.e. (R, m) is a field and we are done in this case also.

n > 0, k > 0: Now, by Lemma 2 above, m cannot consist entirely of zero divisors.

So, pick an $x \in m/m^2$ which is not a zero divisor of R and let $S = R/(x)$. Then

$$\text{Krull dimension } S = (\text{ Krull dimension } R) - 1 \ .$$

Claim: $m/(x)$ is a direct summand of the S-module m/xm.

Before giving the proof of the Claim, let's see how the proof of it will finish off the theorem.

By the Third Change of Rings Theorem, $pd_S(m/xm) = pd_R m = n$. So, if the claim were true, $pd_S(m/(x)) < \infty$ since it is a direct summand of a module with finite projective dimension.

So, we could apply induction to $S = R/(x), \overline{m} = m/(x)$ and conclude that S is a regular local ring and $pd_S(m/(x)) = k - 2$. It follows immediately that (R, m) is a regular local ring. So, the only thing remaining to show would be that

$$pd_R m = 1 + pd_S(m/(x)) \ .$$

Now, by the exact sequence

$$0 \to (x) \to m \to m/(x) \to 0$$

and the fact that (x) is a free R-module and m is <u>not</u> free we obtain that

$$pd_R m = pd_R(m/(x)) \ . \tag{$*$}$$

Now, since $pd_S(m/(x)) < \infty$ we have, by the First Change of Rings Theorem,

$$pd_R(m/(x)) = 1 + pd_S(m/(x))$$

and so, by $(*)$, we are done.

Thus it remains only to prove the Claim.

Proof of Claim: Since $x \notin m^2$, extend x to a minimal generating set of m, call that set $x, y_1, \ldots, y_r$. Let $N = xm + (y_1, \ldots, y_r)$, then $N \subseteq M$ and clearly $N + (x) = m$ since we have a set of generators for m.

Claim: $N \cap (x) = xm$.

Pf: The inclusion $\supseteq$ is obvious.

It remains only to show "$\subseteq$ ". So, let $z \in N \cap (x)$. Then $z = ax = b_1 y_1 + \cdots + b_r y_r + cx$ where $c \in m$ and a, b_i are in R.

Then $ax - b_1 y_1 - \cdots - b_r y_r = cx \in m^2$. So, considering this expression in m/m^2 as an R/m-module, we get that

$$\overline{ax} - \overline{b_1}\overline{y_1} - \cdots - \overline{b_r}\overline{y_r} = 0 \ .$$

But, $\{x, y_1, \ldots, y_r\}$ are a basis for m/m^2 and hence $a, b_1, \ldots, b_r \in m$.

In particular, $z = ax \in mx$ as we wanted to show.

Thus,

$$\frac{m}{xm} = \frac{N + (x)}{N \cap (x)} = \frac{N}{N \cap (x)} + \frac{(x)}{N \cap (x)}$$

and the sum is direct. Thus,

$$\frac{m}{xm} \simeq \frac{N}{N \cap (x)} \oplus \frac{(x)}{N \cap (x)}$$

But

$$\frac{N}{N \cap (x)} \simeq \frac{N + (x)}{(x)} = \frac{m}{(x)}$$

as we wanted to show.

That completes the proof of the Claim and hence of the theorem. □

The last theorem I want to mention is the famous:

Auslander-Buchasbaum Theorem:

Let (R, m) be a noetherian local ring and let A be a finitely generated R-module with $pd_R A < \infty$. Then

$$pd_R A + G(A) = G(R) \ .$$

(Aside: This says that if $pd_R A < \infty$ then $pd_R A \leq G(R) \leq$ Krull dimension R, and this doesn't depend at all on the global dimension of R.)

Proof: Lemma 2 above is (almost) a special case of this theorem. So, suppose that $G(R) = 0$, i.e. m consists entirely of zero-divisors. We showed that if A is finitely generated and $pd_R A < \infty$ then $pd_R A = 0$, i.e. A is free. But, if A is free then $G(A) = G(R)$ and so we are done in this case.

Now suppose that $G(A) = 0$ (and we might as well assume that $G(R) \neq 0$) i.e. every element of m is a zero divisor on A.

But then, as I mentioned earlier, this implies that $m = ann(a)$ for some $a \in A$. So, if we consider an exact sequence

$$0 \to K \to R^n \xrightarrow{\epsilon} A \to 0$$

and let $u \in R^n$ be such that $\epsilon(u) = a$, then $mu \subseteq K$. Since m does not consist entirely of zero divisors we can find an $x \in m$ that is not a divisor of zero in R^n, and hence not a zero divisor on K.

> *Claim:* $G(K/xK) = 0$.
> *Pf:* Let $y \in m$ be arbitrary. We need to show that y is a zero divisor on K/xK.

Consider yu. As we saw above, $yu \in K$ and so $s(yu) \in xK$, i.e. $y\overline{xu} = 0 \in K/xK$. If we can show that $\overline{xu} \neq 0 \in K/xK$ we'll be done.

But $xu \in xK$ implies that $xu = xk$ for some $k \in K$ and so $x(u-k) = 0$ in R^n, i.e. x is a zero divisor on R^n. But, that is a contradiction unless $u = k$. But $u \notin K$ since $\epsilon(u) = a \neq 0$. □

Now, since x is not a zero divisor on K we have (by the 3rd Change of Rings Theorem) that

$$pd_R K = pd_{R/(x)}(K/xK) \ .$$

Since $G(R/(x) = G(R) - 1$ be have, by induction, that

$$pd_{R/(x)}(K/xK) + \big(G_{R/(x)}(K/xK) = 0\big) = G(R/(x)) \ .$$

Thus,

$$pd_R A + G(A) = (1 + pd_R K) + 0 = (1 + pd_{r/(x)}(K/xK)) + 0 =$$

$$(1 + G(R/(x)) = G(R)$$

as we wanted to show. That completes the proof in this special case.

We are now ready for the general proof, which we do by double induction on $G(R)$ and $G(A)$. So, suppose that both of these numbers are not 0.

Claim: There is an $x \in m$ which is not a zero divisor on either R or A.

(This is a standard fact from elementary prime avoidance arguments in commutative algebra, and will be left out.)

Assuming the claim to be true, we have, by the corollary to the Third Change of Rings Theorem that

$$pd_R A + 1 = pd_R(A/xA) \text{ and } pd_R A = pd_{R/(x)}(A/xA)$$

and

$$G(A/xa) + 1 = G(A) \ .$$

So, by induction, applied to the R-module A/xA, we get

$$pd_R(A/xA) = G_R(A/xA) = G(R)$$

which, rewritten, gives

$$(pd_R A + 1) + (G_R(A) - 1) = G(R)$$

which is what we wanted to show. $\square$.