Non Cohen-Macaulay Projective Monomial Curves

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(joint with Les Reid)

$\mathscr{S} = \{a_1, \dots, a_k\}$ a sequence of integers with $0 < a_1 < \cdots < a_k = d$, and $\gcd(a_i) = 1$.

S (or $S_{\mathscr{S}}$) the subsemigroup of $\mathbb{N}^2$ generated by $\{(d,0), (d-a_1, a_1), (d-a_2, a_2), \dots, (d-a_{k-1}, a_{k-1}), (0,d)\}$.

$K[S] \cong K[s^d, s^{d-a_1}t^{a_1}, s^{d-a_2}t^{a_2}, \dots, t^d] \subset K[s,t]$ K a field.

Proj $K[S]$ is a projective monomial curve of degree d.

Question: When is $K[S]$ Cohen-Macaulay?

Some notation: $\deg(s^i t^j) = (i+j)/d$ $(\in \mathbb{N})$

Then $K[S]$ is graded, generated by elements of degree 1.

$\{s^d, t^d\}$ is a homogeneous system of parameters in $K[S]$, and S is Cohen-Macaulay if and only if $\{s^d, t^d\}$ is a regular sequence (that is, t^d is a non-zero-divisor in $K[S]/s^d K[S]$).

Example: Let $\mathscr{S} = \{1, 3, 4\}$, so that $K[S] = K[s^4, s^3t, st^3, t^4]$. We have $s^6t^2 = (s^3t)^2 \in K[S]$, but $s^6t^2 \notin s^4K[S]$ because $s^2t^2 \notin K[S]$. But $(s^6t^2)(t^4) = (s^4)(st^3)^2 \in s^4K[S]$. Therefore t^4 is a zero-divisor in $K[S]/s^4K[S]$, $\{s^4, t^4\}$ is not a regular sequence in $K[S]$, and $K[S]$ is not Cohen-Macaulay.

Example: If $\mathscr{S}$ is an arithmetic progression, e.g. $\mathscr{S} = \{1, 3, 5\}$ then $K[S]$ is Cohen-Macaulay.

More notation: The quotient group of $S_{\mathscr{S}}$ will be denoted $G_{\mathscr{S}}$ (or simply G, or G_d if we wish to indicate what d is) and is equal to $\{(x,y) \in \mathbb{Z}^2 | x+y \equiv 0 \mod d\}$.

$S' = \{\sigma \in G_{\mathscr{S}} | \sigma + a(0,d) \in S$ and $\sigma + b(d,0) \in S$ for some $a, b \geq 0\}$. Then S' is a semigroup and

Theorem (Goto-Suzuki-Watanabe) $K[S]$ is Cohen-Macaulay if and only if $S' = S$.

$\tilde{S} = \{\sigma \in G_{\mathscr{S}} | \sigma + (0,d) \in S$ and $\sigma + (d,0) \in S\}$. Then $S \subseteq \tilde{S} \subseteq S' \subseteq G_{\mathscr{S}} \cap \mathbb{N}^2$. Furthermore (if $S \subsetneq S'$) $\tilde{S} \setminus S$ is the set of maximal elements of $S' \setminus S$ so $S = S'$ if and only if $S = \tilde{S}$. However $\tilde{S}$ need not be a semigroup.

Let T be the subsemigroup of S generated by $(d, 0)$ and $(0, d)$. The spanning set of S over T is defined to be $\mathcal{B} = \{\sigma \in S | \sigma$ cannot be written in the form $\sigma = \sigma' + \tau$ with $\tau \in T,\ \sigma' \in S\}$. A minimal generating set of R as a module over $k[s^d, t^d] \cong k[T]$ is $\{(s^a t^b | (a, b) \in \mathcal{B}\}$, and R is a free $k[s^d, t^d]$ module (hence Cohen-Macaulay) if and only if $\mathcal{B}$ contains d elements. More generally we have

Theorem: The following are equivalent.

1. The spanning set $\mathcal{B}$ of S over T contains $d + a$ elements $(a \geq 0)$.
2. $|\tilde{S} \backslash S| = a$.

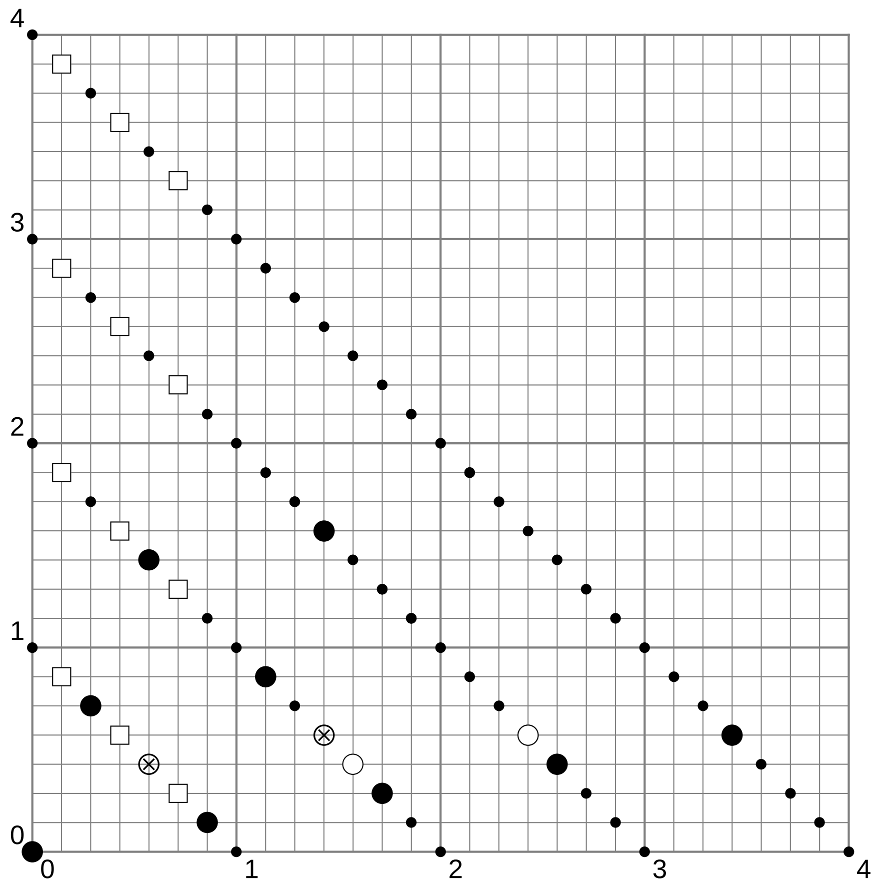
$\mathcal{B}$ is the solid dots in the various diagrams.

Some data:

d	c	CM	nCM	d_1
1	1	1	0	0
2	1	1	0	0
3	3	3	0	0
4	6	5	1	1
5	15	12	3	2
6	27	16	11	10
7	63	37	26	20
8	120	51	69	62
9	252	97	155	123
10	495	142	353	333
11	1023	257	766	663
12	2010	359	1651	1555
13	4095	647	3448	3132
14	8127	920	7207	6966
15	16365	1605	14760	13886
16	32640	2266	30374	29631
17	65535	3795	61740	59431
18	130788	5410	125378	123555

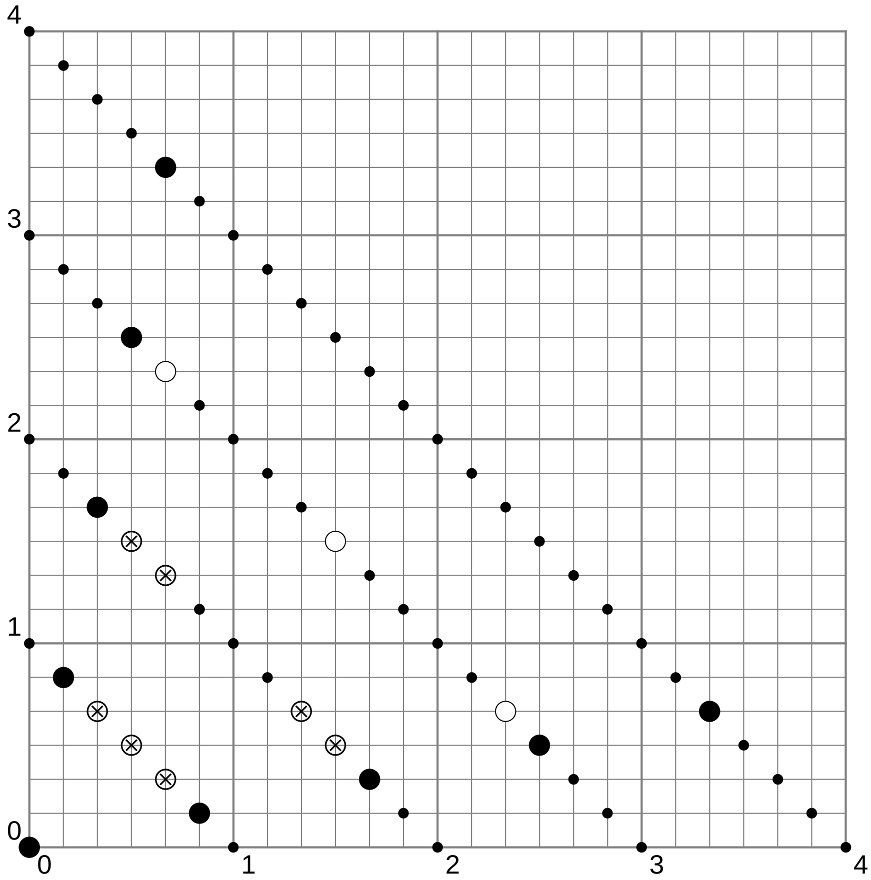
- c total number of curves of degree d
- CM the number of these that are Cohen-Macaulay
- nCM the number that are not Cohen-Macaulay
- d_1 the number that contain an element of $\tilde{S} \backslash S$ of degree 1.

Figure 1: $\mathscr{S} = \{1, 5, 7\}$. The solid dots (large or small) are the elements of $\mathscr{S}$.



- ○ elements of $\tilde{S} \backslash S$
- $\otimes$ the remaining elements of $S' \backslash S$
- □ elements not in S'

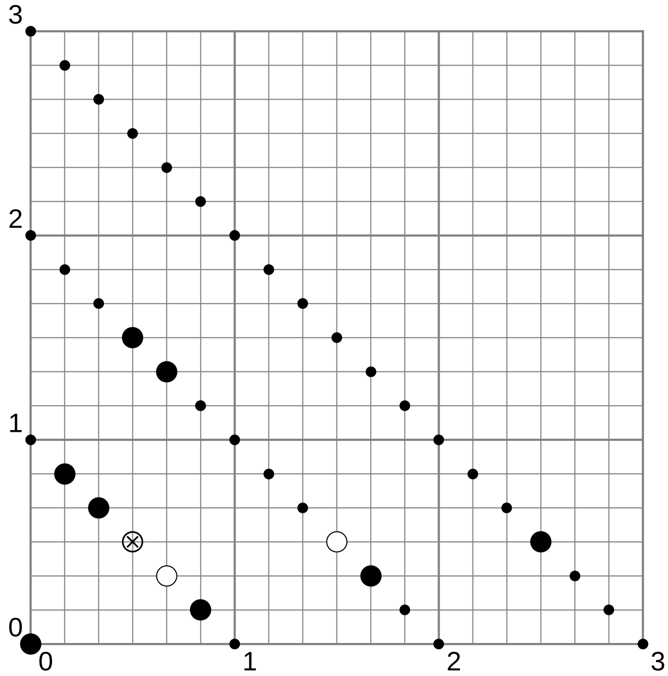
Figure 2: $\mathscr{S} = \{1, 5, 6\}$. As in the previous example the solid dots (large or small) are the elements of $\mathscr{S}$.



○ elements of $\tilde{S}\backslash S$

⊗ the remaining elements of $S'\backslash S$

Figure 3: $\mathscr{S} = \{1, 4, 5, 6\}$. As in the previous examples the solid dots (large or small) are the elements of $\mathscr{S}$.



○ elements of $\tilde{S} \backslash S$

⊗ the remaining element of $S' \backslash S$

Theorem: As the degree d goes to infinity the fraction of all curves of degree d that contain an element of $\tilde{S}\backslash S$ of degree one approaches 1.

None of these are Cohen-Macaulay so we have

Corollary: As the degree d goes to infinity the fraction of all curves of degree d that are Cohen-Macaulay approaches 0.

Approach: The condition that $\tilde{S}\backslash S$ contain an element $(d-x,x)$ of degree 1 is equivalent to the following. There exists $x \in [1, d-1]$ such that $x \notin \mathscr{S}$, $x = a_1 + a_2$,
$d + x = a_3 + a_4$ with $a_i \in \mathscr{S},\ 1 \le i \le 4$. We show that this happens for "most" $\mathscr{S}$.

Illustration: In Figure 3, $\mathscr{S} = \{1,4,5,6\}$, $2 \notin \mathscr{S}$, but $2 = 1+1$ and $6+2 = 8 = 4+4$ with $1 \in \mathscr{S},\ 4 \in \mathscr{S}$ so the above holds with $x = 2,\ a_1 = a_2 = 1, a_3 = a_4 = 4$, corresponding to $(4,2) \in \tilde{S}\backslash S$ of degree 1.

The approach in more detail. (somewhat simplified) For fixed d the sets $\mathscr{S}$ correspond to subsets of $[1, d-1]$. For $x \in [1, d-1]$ let C_x be all subsets of $[1, d-1]$ that do not contain x and let $NC_x \subset C_x$ be all subsets for which a_i, $1 \leq i \leq 4$ can be found, as described above. The cardinality of $|NC_x|$ can be determined explicitly, and $|C_x|$ is approximately equal to 2^{d-2}. From this we were able to give a lower bound to $|\cup NC_x|/|\cup C_x|$, showing that as $d \to \infty$ this approaches 1. The denominator $|\cup C_x|$ represents essentially all curves of degree d and the numerator $|\cup NC_x|$ represents all curves of degree d for which $\tilde{S} \backslash S$ contains an element of degree one.

Definition: Define a curve $\mathscr{S}$ to be maximal if (for the corresponding S and S') $S' \backslash S$ contains no elements of degree one.

The name 'maximal' comes from the following: among all curves $\mathscr{S}$ for which $S_{\mathscr{S}}$ has the same projection onto first and second summands, these are the ones for which the set $\mathscr{S}$ is largest.

Example: The curve $\mathscr{S} = \{1, 5, 7\}$ is not maximal because $S' \backslash S$ contains the element $(4, 3)$ of degree 1. Correspondingly S projects onto the numerical semigroup $\mathbb{N}$ in the second copy of $\mathbb{Z}$ and $\{2, 4, 6 \rightarrow\}$ in the first, as does also the larger curve $\mathscr{S} = \{1, 3, 5, 7\}$, which is maximal (and Cohen-Macaulay).

It is clear that every Cohen-Macaulay curve is maximal. But a maximal curve need not be Cohen-Macaulay, the example of smallest degree being $\mathscr{S} = \{2, 5, 7\}$. Maximal is easier to handle than Cohen-Macaulay. For example, we can prove

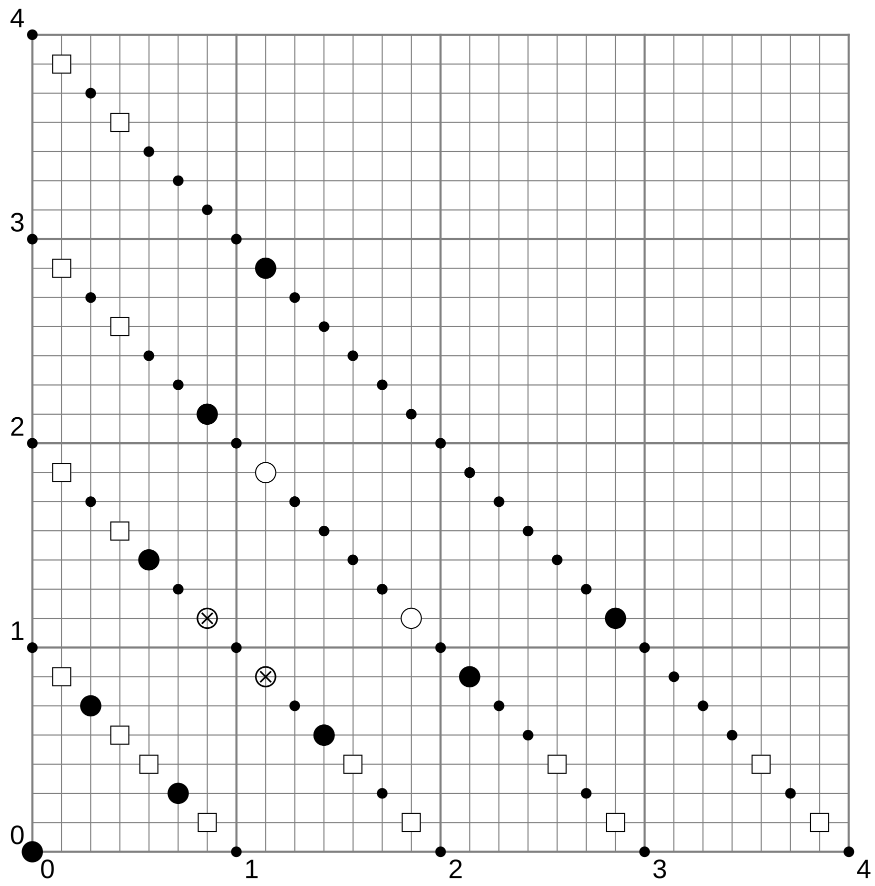
Theorem: Every maximal $\mathscr{S}$ of degree d with $|\mathscr{S}| < d$ is contained in a maximal $\mathscr{S}_1$ of degree d with $|\mathscr{S}_1| = 1 + |\mathscr{S}|$.

We have not been able prove

Conjecture: Every Cohen-Macaulay $\mathscr{S}$ with $|\mathscr{S}| < d$ is contained in a Cohen-Macaulay $\mathscr{S}_1$ with $|\mathscr{S}_1| = 1 + |\mathscr{S}|$.

However the conjecture has been checked in all degrees up to 21.

Figure 4: $\mathscr{S} = \{2, 5, 7\}$. The solid dots (large or small) are the elements of $\mathscr{S}$.



○ elements of $\tilde{S}\backslash S$

$\otimes$ the remaining elements of $S'\backslash S$

The data again, with maximal added:

d	c	CM	nCM	$d1$	e
3	1	1	0	0	1
4	4	3	1	1	3
5	11	8	3	2	8
6	25	14	11	10	14
7	57	31	26	20	32
8	116	47	69	62	48
9	246	91	155	123	100
10	491	138	353	333	142
11	1013	247	766	663	284
12	2006	355	1651	1555	384
13	4083	635	3448	3132	753
14	8121	914	7207	6966	1005
15	16357	1597	14760	13886	1903
16	32632	2258	30374	29631	2548
17	65519	3779	61740	59431	4753
18	130782	5404	125378	123555	6163

- c total number of curves of degree d
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- d_1 the number that contain an element of $\tilde{S} \backslash S$ of degree 1.
- e the number that are maximal

Some easy facts about curves of degree d.

- If $\mathscr{S} \subset [d/2, d]$ then $\mathscr{S}$ is maximal.
- consequently the number of maximal curves of degree d is at least (approximately) $2^{d/2}$.
- If $|\mathscr{S}| > d/2$ and $\mathscr{S}$ is maximal then $\mathscr{S}$ is Cohen-Macaulay.
- consequently every Cohen-Macaulay curve with $d > |\mathscr{S}| > d/2$ is contained in a Cohen-Macaulay curve $\mathscr{S}'$ with $|\mathscr{S}'| = |\mathscr{S}| + 1$.

A closer look at the asymptotics. Asymptotically

- The total number of curves of degree d grows as 2^d.
- The number of Cohen-Macaulay curves of degree d grows at least as $2^{.5d}$. (Seen by refining argument of previous slide for maximal curves.)
- The number of maximal curves of degree d grows at most as $2^{.906d}$. (Shown by carefully keeping track of our original argument that most curves contain an element of degree 1 in $\tilde{S}\backslash S$.)

Question: How do the asymptotics of maximal curves compare with those of Cohen-Macaulay curves?