Concluding Lectures

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I gave several concluding lectures to finish our reading course. Some of the motivation for these lectures was

1. To prove several results that we did not get to, especially graded versions of things like the Auslander Buchsbaum theorem (Theorem 13).
2. We discussed local and graded Cohen-Macaulay rings independently of each other, to a large extent. I attempt to unify the two cases. This is done by giving a definition of Cohen-Macaulay for arbitrary Noetherian rings which includes both the local and graded cases.
3. To give other ways of looking at Cohen-Macaulay, besides the definitions in terms of Ext or regular sequences (Theorem 16).
4. I would like to answer (1), (2), and (3) by deriving the graded result from the corresponding local result.

References are easier to come by for the local case. For example, everything that we need to know about local Cohen-Macaulay rings is in [2], who also have some, but not all, of the graded results that we are after. And [5] does talk about non-local Cohen-Macaulay rings, but does not contain the sort of graded result that we want. In what follows I will freely use local results, and try to derive the desired graded results from them. When the reading course was almost finished I came across several references on Invariant Theory ([1], [4],[6]) which discuss the graded case as well as the local case. If these had been followed from the start we might have been able to do the two cases in parallel so this reduction would not be necessary. But having studied the local case one might prefer to try deriving the graded from the local rather than starting over again.

First recall a few facts from the local case. Let $(A, \mathfrak{m})$ be a Noetherian local ring, and $k = R/\mathfrak{m}$. We have seen that all maximal regular sequences of elements of $\mathfrak{m}$ have the same length. This number is called the depth of A, and is denoted $\operatorname{depth}(A)$. One always has

$$\operatorname{depth}(A) \leq \text{Krull } \dim(A)$$

and A is defined to be Cohen-Macaulay if

$$\operatorname{depth}(A) = \text{Krull } \dim(A).$$

More generally if M is any finitely generated A-module we can talk about regular M-sequences (i.e. sequences $x_1, \ldots, x_d$ of elements of $\mathfrak{m}$ such that x_i is a non-zero-divisor on $M/(x_1, \ldots, x_{i-1})M$, $1 \leq i \leq d$.) Then one shows that all maximal regular M-sequences have the same length, which is called $\operatorname{depth}(M)$. Bruns and Herzog [2, Theorem 1.2.5] do this by showing that

$$\operatorname{depth}(M) = \min\{i | \operatorname{Ext}^i_A(k, M) \neq 0\}$$

(the right hand side not depending on the regular sequence).

The approach to (2) [2, page 56] is as follows: take the local case as the primary definition of Cohen-Macaulay. Then if R is an arbitrary Noetherian ring we *define* R to be Cohen-Macaulay if and only if $R_{\mathfrak{m}}$ is a Cohen-Macaulay local ring for all maximal ideals $\mathfrak{m}$ of R. The problem then becomes to show that this notion of Cohen-Macaulay agrees with our graded version.

Now consider (3). If $(A, \mathfrak{m})$ is a Noetherian local ring of Krull dimension n then a system of parameters $x_1, \ldots, x_n$ is a set of elements $\mathfrak{m}$ such that equivalently [2, page 367] $\sqrt{(x_1, \ldots, x_n)} = \mathfrak{m}$ or Krull $\dim(A/(x_1, \ldots, x_n)A) = 0$. A key property of Cohen-Macaulay rings is the following [2, Theorem 2.1.2]:

Theorem 1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then A is Cohen-Macaulay if and only if every system of parameters is a regular sequence.*

A Noetherian local ring $(A, \mathfrak{m})$ is called regular if it has a system of parameters that generates $\mathfrak{m}$. (Bruns and Herzog, Definition 2.2.1, page 64). If $n =$ Krull $\dim(A)$ this is obviously equivalent to $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = n$ or that $\mathfrak{m}$ can be generated by n elements. A substantial theorem is that $(A, \mathfrak{m})$ is regular if and only if $\mathrm{pd}_A(M) < \infty$ for all A-modules M.

An alternative criterion for Cohen-Macaulayness in the local case is the following [2, 2.2.11]:

Theorem 2. *Let A be a Noetherian local ring and B a regular local subring such that A is a finitely generated B-module. Then A is Cohen-Macaulay if and only if A is a free B-module.*

The graded version of this will be stated later.

First of all I will quote an exercise from Bruns and Herzog which is part of the answer to (2). This uses Cohen-Macaulay modules, so I will first define these:

Definition 3. [2, 2.2.1] Let $(A, \mathfrak{m})$ be a Noetherian local ring and $M \neq 0$ a finitely generated A-module. Then

1. $\dim(M) =$ Krull $\dim(A/\mathrm{Ann}(M))$
2. M is a Cohen-Macaulay A-module if $\dim(M) = \mathrm{depth}(M)$.
3. If R is an arbitrary Noetherian ring and M is a finitely generated R-module then M is defined to be a Cohen-Macaulay R-module if $M_{\mathfrak{m}}$ is a Cohen-Macaulay $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of $\mathrm{Supp}(M)$.

Remark 4. Note that R is a Cohen-Macaulay ring if and only if R is Cohen-Macaulay as an R-module.

Theorem 5. [2, 2.1.3] *If M is a Cohen-Macaulay R-module then M_S is a Cohen-Macaulay R_S-module for every multiplicative set S in R. ('stability of the Cohen-Macaulay property under localization').*

In particular, if R is Cohen-Macaulay, then $R_{\mathfrak{p}}$ is Cohen-Macaulay for all prime ideals $\mathfrak{p}$ of R (not just for all maximal ideals).

Recall also the following definitions:

Definition 6.
1. If $\mathfrak{p} \in \operatorname{Spec}(R)$ then $\mathfrak{p}^*$ is the prime ideal generated by all homogeneous elements of $\mathfrak{p}$.
2. If $\mathfrak{p}$ is a graded prime ideal of R then $M_{(\mathfrak{p})}$ is the localization of M with respect to all the homogeneous elements of R that are not in $\mathfrak{p}$. ([2, page30]). (This notation is unfortunate, in that in the theory of schemes $M_{(\mathfrak{p})}$ is used to denote something different, namely the degree 0 part of our $M_{(\mathfrak{p})}$). Then $M_{(\mathfrak{p})}$ is an $R_{(\mathfrak{p})}$-module, and $M_{\mathfrak{p}}$ is a further localization of $M_{(\mathfrak{p})}$.
3. A graded ring is *local if it has a unique maximal homogeneous ideal. [2, 1.5.13].

With these definitions we can now state the Bruns and Herzog exercise:

Exercise 2.1.27 Bruns and Herzog, page 63.

Let R be a Noetherian graded ring and M a finitely generated graded R-module. Show that

(a) For $\mathfrak{p} \in \operatorname{Spec}(R)$ the localization $M_{\mathfrak{p}}$ is a Cohen-Macaulay $R_{\mathfrak{p}}$-module if and only if $M_{\mathfrak{p}^*}$ is a Cohen-Macaulay $R_{\mathfrak{p}^*}$-module.

(b) The following are equivalent:
   (i) M is a Cohen-Macaulay R-module.
   (ii) $M_{\mathfrak{p}}$ is a Cohen-Macaulay $R_{\mathfrak{p}}$-module for all graded prime ideals $\mathfrak{p}$.
   (iii) $M_{(\mathfrak{p})}$ is a Cohen-Macaulay $R_{(\mathfrak{p})}$- module for all graded prime ideals $\mathfrak{p}$.

(c) Suppose in addition that $(R, \mathfrak{m})$ is *local. Then M is a Cohen-Macaulay R-module if and only if $M_{\mathfrak{m}}$ is a Cohen-Macaulay $R_{\mathfrak{m}}$-module.

Remark 7. If R is a finitely generated graded k algebra in non-negative degrees, with $R_0 = k$ (a field) and $\mathfrak{m} = \oplus_{i>0} R_i$ then $(R, \mathfrak{m})$ is *local. This is our usual case. If we apply the above exercise here with $M = R$ we conclude that R is Cohen-Macaulay (in the sense of Bruns and Herzog) if and only if $R_{\mathfrak{m}}$ is Cohen-Macaulay.

Sketch of proof of exercise 2.1.27:

(a) This follows from Theorems 1.5.8 and 1.5.9 of Bruns and Herzog. The first says that if $\mathfrak{p}$ is not graded (so that $\mathfrak{p}^* \subsetneq \mathfrak{p}$) then $\dim M_{\mathfrak{p}} = 1 + \dim M_{\mathfrak{p}}^*$. The second says that $\operatorname{depth} M_{\mathfrak{p}} = 1 + \operatorname{depth} M_{\mathfrak{p}}^*$, from which (a) is immediate.

(b) We have (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) by Theorem 5 because in each case we have a localization. So assume (a) and (ii). Assume that $\mathfrak{p}$ is a maximal ideal of R. Then by (a), $M_{\mathfrak{p}}$ is a Cohen-Macaulay $R_{\mathfrak{p}}$-module if and only if $M_{\mathfrak{p}^*}$ is a Cohen-Macaulay $R_{\mathfrak{p}^*}$ module. But $M_{\mathfrak{p}^*}$ is Cohen-Macaulay by assumption (ii). Therefore $M_{\mathfrak{p}}$ is a Cohen-Macaulay $R_{\mathfrak{p}}$ for all maximal ideals $\mathfrak{p}$ of R, so M is a Cohen-Macaulay R-module by definition.

(c) Suppose that $M_{\mathfrak{m}}$ is a Cohen-Macaulay $R_{\mathfrak{m}}$-module. Let $\mathfrak{p}$ be any graded prime ideal of R. Then, since R is *local, $\mathfrak{p} \subseteq \mathfrak{m}$ so $M_{\mathfrak{p}}$ is a further localization of $M_{\mathfrak{m}}$.

Therefore $M_{\mathfrak{p}}$ is also Cohen-Macaulay. Thus by (b)(ii) M is a Cohen-Macaulay R-module. The other implication follows from Theorem 5.

Let R be a graded ring in non-negative degrees, with $R_0 = k$, a field, and $\mathfrak{m} = \oplus_{i>0} R_i$. We relate properties of R to properties of $R_{\mathfrak{m}}$, assuming the exercise, which implies that R is Cohen-Macaulay (as defined by Bruns and Herzog) if and only if $R_{\mathfrak{m}}$ is Cohen-Macaulay.

Let M be a finitely generated graded R-module. Then by Theorem 1.2.5 page 10 of Bruns and Herzog (discussed in our recent Homological Algebra course, and by Brian) we have that $\min\{i|\mathrm{Ext}^i_R(k, M) \neq 0\}$ is equal to the length of all maximal M-regular sequences in $\mathfrak{m}$. We have seen in Corollary 1.24 of Susan's lectures (based on page 34 of Bruns and Herzog) that $\min\{i|\mathrm{Ext}^i_R(k, M) \neq 0\}$ is also the length of all maximal M-sequences consisting of homogeneous elements of R of positive degree.

Now observe that $\mathrm{Ext}^i_R(k, M)$ is killed by $\mathfrak{m}$ (because of the k in the first position). Any element $r \in R\backslash\mathfrak{m}$ is of the form $\lambda + x$ with $\lambda \in k^*$, $x \in \mathfrak{m}$. This acts on M in the same way that λ does, i.e. invertably. Therefore

$$\mathrm{Ext}^i_R(k, M) \otimes_R R_{\mathfrak{m}} \cong \mathrm{Ext}^i_R(k, M)$$

(as abelian groups or as R-modules). By [7, Proposition 3.3.10] we have

$$\mathrm{Ext}^i_R(k, M) \otimes_R R_{\mathfrak{m}} \cong \mathrm{Ext}^i_{R_{\mathfrak{m}}}(k \otimes_R R_{\mathfrak{m}}, M_{\mathfrak{m}})$$

and since $k \otimes_R R_{\mathfrak{m}} \cong k$ (again because $R\backslash\mathfrak{m}$ acts invertibly on k) we have

$$\mathrm{Ext}^i_{R_{\mathfrak{m}}}(k, M_{\mathfrak{m}}) \cong \mathrm{Ext}^i_R(k, M)$$

hence

$$\min\{i|\mathrm{Ext}^i_R(k, M) \neq 0\} = \min\{i|\mathrm{Ext}^i_{R_{\mathfrak{m}}}(k, M_{\mathfrak{m}})\}.$$

Definition 8. Let R be a graded ring in non-negative degrees, with $R_0 = k$, a field, and $\mathfrak{m} = \oplus_{i>0} R_i$. Let M be a finitely generated graded R-module. Then $\mathrm{depth}_R M$ is the length of all maximal M-regular sequences in $\mathfrak{m}$ consisting of homogeneous elements of positive degree.

By the above discussion the left hand side of the last equality is $\mathrm{depth}_R M$. By the local case the right hand side is $\mathrm{depth}_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$, i.e. the length of all maximal $M_{\mathfrak{m}}$-regular sequences in $\mathfrak{m}R_{\mathfrak{m}}$. Under the same hypotheses as Definition 8 we have

Theorem 9. $\mathrm{depth}_R M = \mathrm{depth}_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$

Taking $M = R$ (noting that Krull $\dim(R) =$ Krull $\dim(R_{\mathfrak{m}})$) and putting all the above together we conclude (again under the hypotheses of Definition 8)

Theorem 10. *R is Cohen-Macaulay in the sense of Bruns and Herzog if and only if R has a regular sequence of length* Krull $\dim(R)$ *consisting of homogeneous elements of positive degree.*

Now we consider the graded Auslander-Buchsbaum theorem. Let R continue to be a finitely generated graded k-algebra in non-negative degrees, with $R_0 = k$, and let M be a finitely generated graded R-module. Then we have

Lemma 11. *$M = 0 \Leftrightarrow M_{\mathfrak{m}} = 0$.*

Proof: The implication $\Rightarrow$ is obvious. In order to prove the other implication assume that $M \neq 0$. Then M has an associated prime $\mathfrak{p}$, which is homogeneous, hence contained in $\mathfrak{m}$. The associated primes of M are contained in $\mathrm{Supp}(M)$ so $M_{\mathfrak{p}} \neq 0$, hence $M_{\mathfrak{m}} \neq 0$.

Lemma 12. *With the same hypotheses as in Definition 8, $\mathrm{pd}_R M = \mathrm{pd}_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$.*

Proof: Let $\mathrm{pd}_R(M) = d$. This means that M has a projective resolution of length d. We can choose the resolution to be by finitely generated graded projective R-modules. But finitely generated graded projective R-modules are free, so we have a resolution

$$0 \to R^{n_d} \xrightarrow{f_d} \cdots \to R^{n_1} \to R^{n_0} \to M \to 0$$

Since $\otimes_R R_{\mathfrak{m}}$ is exact we have an exact sequence

$$0 \to R_{\mathfrak{m}}^{n_d} \xrightarrow{f_d} \cdots \to R_{\mathfrak{m}}^{n_1} \to R_{\mathfrak{m}}^{n_0} \to M_{\mathfrak{m}} \to 0.$$

From this it follows that $\mathrm{pd}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \leq \mathrm{pd}_R M$. Now suppose that $\mathrm{pd}_{R_m} M_{\mathfrak{m}} < d$. Then $\ker(R_{\mathfrak{m}}^{n_{d-2}} \underset{f_{d-2}\otimes 1}{\to} M_{\mathfrak{m}}^{n_{d-3}}) \cong (\ker f_{d-2}) \otimes R_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module. Thus we are reduced to showing that if N is a finitely generated graded R-module then $N_{\mathfrak{m}}$ a free $R_{\mathfrak{m}}$-module implies N is a free R-module.

This can be seen as follows. Suppose that N has a minimal homogeneous generating set $n_1, \ldots, n_\ell$. Then $n_1 \otimes 1, \ldots, n_\ell \otimes 1$ is a generating set for $N_{\mathfrak{m}}$, hence $\overline{n_1 \otimes 1}, \ldots, \overline{n_\ell \otimes 1}$ is a generating set for $N_{\mathfrak{m}}/\mathfrak{m}N_{\mathfrak{m}} \cong N/\mathfrak{m}N$ over $R/\mathfrak{m}$ (for the last isomorphism note that $R\backslash\mathfrak{m}$ acts invertibly on $N/\mathfrak{m}N$ so we have $N/\mathfrak{m}N \otimes_R R_{\mathfrak{m}}$ and the latter is isomorphic to $N_{\mathfrak{m}}/\mathfrak{m}N_{\mathfrak{m}}$ by properties of localization). Some subset of $\overline{n_1 \otimes 1}, \ldots, \overline{n_\ell \otimes 1}$ is a k-basis of $N/\mathfrak{m}N$, hence the corresponding subset of $n_1 \ldots, n_\ell$ generates N by the graded Nakayama lemma. So assume $\overline{n_1 \otimes 1}, \ldots, \overline{n_\ell \otimes 1}$ is a basis of $N/\mathfrak{m}N$ and consider the exact sequence

$$0 \to X \to R^\ell \to N \to 0$$

where the surjection $R^\ell \to N$ takes the i^{th} standard basis vector e_i to n_i. Then $\otimes_R R_{\mathfrak{m}}$ yields an exact sequence

$$0 \to X_{\mathfrak{m}} \to R_{\mathfrak{m}}^\ell \to N_{\mathfrak{m}} \to 0.$$

Since $N_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$ module the last sequence splits. Hence

$$0 \to \frac{X_{\mathfrak{m}}}{\mathfrak{m}X_{\mathfrak{m}}} \to \left(\frac{R_{\mathfrak{m}}}{\mathfrak{m}R_{\mathfrak{m}}}\right)^\ell \underset{\cong}{\to} \frac{N_{\mathfrak{m}}}{\mathfrak{m}N_{\mathfrak{m}}} \to 0$$

is exact. Hence $X_\mathfrak{m}/\mathfrak{m}X_\mathfrak{m} = 0$, so $X_\mathfrak{m} = 0$ by the usual Nakayama lemma, which implies that $X = 0$ by Lemma 11 (recall that X is graded). Hence $R^\ell \cong N$, as we wished to show.

The Auslander-Buchsbaum theorem for $M_\mathfrak{m}$ as an $R_\mathfrak{m}$-module says that (if $\mathrm{pd}_{R_\mathfrak{m}} M_\mathfrak{m} < \infty$)

$$\mathrm{pd}_{R_\mathfrak{m}} M_\mathfrak{m} + \mathrm{depth}_{R_\mathfrak{m}} M_\mathfrak{m} = \mathrm{depth} R_\mathfrak{m}$$

(Theorem 1.3.3 of Bruns and Herzog). But $\mathrm{pd}_{R_\mathfrak{m}} M_\mathfrak{m} = \mathrm{pd}_R M$ by Lemma 12 and $\mathrm{depth}_{R_\mathfrak{m}} M_\mathfrak{m} = \mathrm{depth}_R M$, $\mathrm{depth} R_\mathfrak{m} = \mathrm{depth}_{R_\mathfrak{m}} R_\mathfrak{m} = \mathrm{depth}_R R = \mathrm{depth} R$ by Theorem 9.

Making these substitutions we obtain

Theorem 13. *(graded Auslander Buchsbaum) Let R be a finitely generated graded k-algebra in non-negative degrees with $R_0 = k$ a field. Let M be a finitely generated graded R-module with $\mathrm{pd}_R M < \infty$. Then*

$$\mathrm{pd}_R M + \mathrm{depth}_R M = \mathrm{depth} R$$

Corollary 14. *Let $R = k[x_0, \dots, x_n]$ and I a homogeneous ideal in R. Then R/I is Cohen-Macaulay if and only if $\mathrm{pd}_R R/I = n + 1 - \text{Krull dim} R/I$.*

This is an easy way of deciding if R/I is Cohen-Macaulay because $\mathrm{pd}_R R/I$ and Krull $\dim R/I$ can both be found with a computer algebra system. In particular if R/I is the homogeneous co-ordinate ring of a curve then $\mathrm{depth}_R R/I = 1$ or 2 and Krull $\dim R/I = 2$ so $\mathrm{pd}_R R/I = n$ or $n - 1$ and R/I is Cohen-Macaulay if and only if $\mathrm{pd}_R R/I = n - 1$.

Now let R be a graded k-algebra in degrees ≥ 0, as above. Let $n = \text{Krull dim} R$. A set $\{f_1, \dots, f_n\} \subset \mathfrak{m}$ of homogeneous elements of positive degree is called a homogeneous system of parameters if equivalently

1. R is finitely generated as a module over $k[f_1, \dots, f_n]$ or
2. $\dim_k R/(f_1, \dots, f_n)R < \infty$ ($\Leftrightarrow$ Krull $\dim R/(f_1, \dots f_n)R = 0$).

Theorem 15. *If R is a finitely generated graded k-algebra then a homogeneous system of parameters exists.*

A proof can be found in [6] (page 5 in the numbering at the top of the page). On the previous page the equivalence of (1) and (2) above is discussed, where it is also observed that the elements of a homogeneous system of parameters are independent indeterminates over k.

Now suppose that $R = \oplus_{i \geq 0} R_i$ is a finitely generated graded $R_0 = k$ algebra, k a field, and $B = k[f_1, \dots, f_n]$ where $f_1, \dots, f_n$ is a homogeneous system of parameters. Applying Theorem 13 to the finitely generated B-module R we obtain

$$\mathrm{pd}_B R + \mathrm{depth}_B R = \mathrm{depth} B \ (= n).$$

Note that $\mathrm{pd}_B R < \infty$ because B is a polynomial ring, hence regular. Suppose that R is Cohen-Macaulay. Then every homogeneous system of parameters is regular. (This will be proved later.) In particular $f_1, \dots f_n$ is regular so $\mathrm{depth}_B R \geq n$. But R is integral over B so Krull $\dim R =$ Krull $\dim B = n$. Hence $\mathrm{depth}_B R \leq \mathrm{depth}_R R \leq n$, so $\mathrm{depth}_B R = n$. By the graded Auslander Buchsbaum Theorem we conclude that $\mathrm{pd}_B R = 0$, and hence that R is a projective B-module. But graded projective B-modules are free, so R is a free B-module. (The proof that graded projective B-modules are free (with a homogeneous basis) is an easy graded Nakayama type argument similar to that used in the proof of Lemma 12, or follows from [3, Chapter VIII, Theorem 6.1].

Conversely if R is a free B-module then obviously $f_1, \dots, f_n$ is a regular sequence in R so we conclude that R is Cohen-Macaulay. This proves (modulo the discussion of homogeneous systems of parameters in Lemma 17 below)

Theorem 16. *Let R be a finitely generated graded k-algebra in non-negative degrees over a field k, with $R_0 = k$. Then R is Cohen-Macaulay if and only if R is a free module over $k[f_1, \dots, f_n]$ for any (equivalently every) homogeneous system of parameters of R.*

Lemma 17. *Let R be a finitely generated graded k-algebra in non-negative degrees over a field k, with $R_0 = k$. Then every homogeneous system of parameters in R is a regular sequence in R.*

Proof: We have the easy implications

$$f_1, \dots, f_n \text{ is a h.s.o.p. in } R \Leftrightarrow \dim_k \frac{R}{(f_1, \dots f_n)} < \infty \Rightarrow$$

$$\dim_k \frac{R_{\mathfrak{m}}}{(f_1, \dots f_n)R_{\mathfrak{m}}} = 0 \Rightarrow f_1, \dots, f_n \text{ is a s.o.p. in } R_m \Rightarrow$$

$$f_1, \dots, f_n \text{ is a regular sequence in } R_{\mathfrak{m}}.$$

The last implication follows from the local case. It now suffices to prove that a sequence of homogeneous elements $f_1, \dots f_n$ in R of positive degree which form a regular sequence in $R_{\mathfrak{m}}$ is already a regular sequence in R. Suppose that $f_1 x = 0$ in R. By looking at homogeneous components we may assume that x is homogeneous. Then $f_1 x = 0$ in $R_{\mathfrak{m}}$ which implies that $x = 0$ in $R_{\mathfrak{m}}$. Thus there exists an element $y \in R \backslash \mathfrak{m}$ such that $yx = 0$ in R. Looking at homogeneous components we conclude that $x = 0$ in R, i.e. that f_1 is a non-zero-divisor in R. Let $\overline{\mathfrak{m}}$ be the image of $\mathfrak{m}$ in the graded ring $R/f_1 R$ ($\overline{\mathfrak{m}}$ is then the ideal generated by all homogeneous elements of positive degree in $R/f_1 R$). We have

$$\left(\frac{R}{f_1 R}\right)_{\overline{\mathfrak{m}}} \cong \frac{R}{f_1 R} \otimes_R R_{\mathfrak{m}} \cong \frac{R_{\mathfrak{m}}}{f_1 R_{\mathfrak{m}}}$$

It now follows by induction applied to the rings $R/f_1 R$, $(R/f_1 R)_{\overline{\mathfrak{m}}}$, and the sequence $f_2, \dots, f_n$ (which is a regular sequence in $R_{\mathfrak{m}}/f_1 R_{\mathfrak{m}}$, hence in $(R/f_1 R)_{\overline{\mathfrak{m}}}$) that $f_1, \dots f_n$ is a regular sequence in R. This proves

I will conclude with two examples which show that one has to be careful when discussing the concept of Cohen-Macaulay in other than the local or graded cases. If R is any commutative Noetherian ring one can still define a sequence $f_1, \dots, f_n$ to be regular in R in the usual way: f_i is a non-zero-divisor in $R/(f_1, \dots, f_{i-1})R$, $1 \leq i \leq n$ and $(f_1, \dots, f_n)R \neq R$. (The second assumption was automatic in the cases we discussed previously.) Furthermore we can define a regular sequence to be maximal if it cannot be extended to a longer regular seequence.

Example 18. Let $R = k[x, y]$ where k is an algebraically closed field. Let S be the multiplicative set generated by all elements $x - a, y - b$ where $a \neq 0, b \neq 0$. Spec$R$ consists of

(i) height 2 prime ideals, which are of the form $(x - \alpha, y - \beta)$, $\alpha \in k$, $\beta \in k$.
(ii) height 1 prime ideas, which are of the form (f), where f is an irreducible polynomial
(iii) (0)

Spec(R_S) is the subset of Spec(R) consisting of all primes that do not contain some element of S. Of type (i) only (x, y) is left. Of type (ii) all are left, except for $(x - \alpha)$ and $(y - \beta)$ where $\alpha \neq 0$ and $\beta \neq 0$. If f is irreducible, not a k^*-multiple of $(x - \alpha)$ or $(y - \beta)$ where $\alpha \neq 0$, $\beta \neq 0$ and $f(0, 0) \neq 0$ then fR_S is a maximal ideal of R_S. This implies that $\{f\}$ is a maximal regular sequence in R_S. On the other hand $\{x, y\}$ is a maximal regular sequence in R_S. The ring R_S is a localization of the Cohen-Macaulay ring R, hence Cohen-Macaulay by Theorem 5. This shows that in a Cohen-Macaulay ring there can exist maximal regular sequences of different lengths. Some authors require that in order for a ring to be Cohen-Macaulay all maximal ideals must have the same height. This avoids the present type of example, but then one would have to be more careful formulating Theorem 5.

Example 19. Let $R = k[x^4, x^3y, xy^3, y^4]$ (graded by giving the algebra generators each degree 1). The Hilbert function of R is 1 4 9 13 17 ... with difference sequence 1 3 5 4 4 $\rightarrow$ from which it is immediate that x^4 cannot be extended to a homogeneous regular sequence of length 2. Therefore R is not Cohen-Macaulay by Theorem 10. However any element in $R \backslash k$ with non-zero constant term is a non-unit, non-zero-divisor in R/x^4R. Therefore R does have maximal regular sequences of length 2 (= Krull dim$R$) (but of course no regular sequence of length 2 can consist of homogeneous elements of positive degree).

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