Type vectors and Hilbert functions

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In these notes I try to understand type vectors in my own way, because Susan (Cooper) has been working with type vectors. The goal is to show [2] that type vectors are in one-to-one correspondence with the Hilbert functions of finite sets of points in $\mathbb{P}^n$ in as conceptual a way as I can.

Let $R = k[x_1, \ldots, x_n]$.

(1) An order ideal of R is a non-empty set M of monomials in $x_1, \ldots, x_n$ such that if $x \in M$ and y is a monomial dividing x then $y \in M$. (Hence $1 \in M$.)

(2) Let $\mathscr{M}$ be the set of all monomials in $x_1, \ldots, x_n$. Then M is an order ideal of R if and only if $\mathscr{M} \backslash M$ is the set of monomials in a monomial ideal of R (easy observation).

(3) If I is any homogeneous ideal of R then R/I has a k-basis which is an order ideal. [5, Theorem 2.1, page 59] The proof is a few paragraphs, so this result, although non-trivial, is not deep like Macaulay's Theorem (a statement of which can be found in [5, Theorem 2.2]).

(3$'$) From (3) it follows that there is a monomial ideal J of R such that R/J has the same Hilbert function as R/I.

(4) Now I recall the definition of deg-rev-lex ordering on $\mathscr{M}$. Let $x^{\boldsymbol{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and $x^{\boldsymbol{\beta}} = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$ where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_n)$ Then $x^\alpha > x^\beta$ means that either $\sum \alpha_i > \sum \beta_i$ (i.e. $\deg x^\alpha > \deg x^\beta$) or $\sum \alpha_i = \sum \beta_i$ and the last non-zero co-ordinate of $\boldsymbol{\alpha} - \boldsymbol{\beta}$ is negative. In the following $>$ will always denote deg-rev-lex order. When comparing monomials of the same degree I will often simply write rev-lex order.

Consequences:

(a) $x_1 > x_2 > \cdots > x_n$

(b) Any monomial containing only $x_1, \ldots, x_{i-1}$ is larger than any monomial of the same degree that contains x_i. (where "contains" means that the variable in question occurs with a positive exponent).

Example: The monomials of degree 2 in decreasing rev-lex order are (largest) $\quad x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3,\ x_2x_3,\ x_3^2 \quad$ (smallest)

These can be grouped in several ways which respect this order:

$\{x_1^2\}$ containing x_1 alone $\subset \{x_1^2, x_1x_2, x_2^2\}$ containing x_1 and x_2 $\subset$ $\{x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2\}$.

or they can be grouped according to the power of x_3 that they contain:

$\{x_1^2, x_1x_2, x_2^2\}$ containing x_3^0 followed by $\{x_1x_3, x_2x_3\}$ containing x_3^1 and finally by $\{x_3^2\}$ containing x_3^2.

An initial rev-lex segment of degree d is a collection $\mathscr{S}$ of monomials of degree d such that if $M \in \mathscr{S}$ and M_1 is a monomial of degree d such that $M_1 > M$ then $M_1 \in \mathscr{S}$. That is $\mathscr{S}$ consists of the largest $\#\mathscr{S}$ elements of degree d in rev-lex order. For example $\{x_1^2\}$, $\{x_1^2, x_1x_2\}$ $\{x_1^2, x_1x_2, x_2^2\}$, $\{x_1^2, x_1x_2, x_2^2, x_1x_3\}$ are all initial rev-lex segments, but $\{x_1^2, x_1x_2, x_1x_3\}$ is not.

(5) **Definition:** A rev-lex segment order ideal M is an order ideal such that for all d the monomials $M_d \subset M$ of degree d are an initial rev-lex segment. (Perhaps I should say *initial* rev-lex segment order ideal, but that is a bit cumbersome. Perhaps there is an established name for these things somewhere in the literature, I haven't looked very hard.)

Write $M(\ell)$ for a rev-lex segment order ideal in ℓ variables $x_1, \ldots x_\ell$ ($\ell \leq n$). (Note that all ℓ variables need not occur in monomials of $M(\ell)$.)

Macaulay's Theorem gives a one-to-one correspondence between Hilbert functions of quotients $k[x_1, \ldots, x_n]/I$ (I a homogeneous ideal) and rev-lex segment order ideals. This goes as follows: to the sequence $1, c_1, c_2, \ldots, c_i \ldots$, ($c_1 \leq n$) define a set $M(n)$ by letting the degree d part $M(n)_d$ be the largest c_d monomials of degree d in $x_1, \ldots, x_n$. The (highly non-trivial) theorem is that $1, c_1, c_2, \ldots, c_i \ldots$ is the Hilbert function of a quotient of $k[x_1, \ldots, x_n]/I$ by a homogeneous ideal if and only if $M(n)$ is an order ideal. In this correspondence zero dimensional quotients of $k[x_1, \ldots, x_n]$ correspond to finite order ideals.

(6) Let $M(n)$ be any order ideal of $R = k[x_1, \ldots, x_n]$. Let $M_i(n-1) = \{x/x_n^i | x \in M(n)$ and x is divisible by x_n^i but no higher power of $x_n\}$. Then we have a set theoretical disjoint union

$$M(n) = M_0(n-1) \cup x_n M_1(n-1) \cup x_n^2 M_2(n-1) \cup \cdots \cup x_n^r M_r(n-1)$$

for some $r \geq 0$. It is easily checked that $M_i(n-1)$ is an order ideal in $k[x_1, \ldots, x_{n-1}]$. Indeed, if $x \in M_i(n-1)$ and y is a monomial in $k[x_1, \ldots, x_{n-1}]$

dividing x then we have $x_n^i x \in M(n)$. Furthermore $yx_n^i | x_n^i x$ so $yx_n^i \in M(n)$. But y contains no factor x_n so $y \in M_i(n-1)$, as claimed. Note that in the ordering of $M(n)$ each element of $[x_n^{i-1} M_{i-1}(n-1)]_d$ is larger than every element of $[x_n^i M_i(n-1)]_d$, and that within $[x_n^i M_i(n-1)]_d$ the ordering is given by that of $M_i(n-1)_{d-i}$. That is,

$$x_n^i x > x_n^i y \text{ in } [x_n^i M_i(n-1)]_d \text{ (or in } M(n)) \Leftrightarrow x > y \text{ in } M_i(n-1)_{d-i}$$

From this it is clear that if $M(n)$ is a rev-lex segment order ideal, so is each $M_i(n-1)$. Furthermore, by the definition of rev-lex order

$$[x_n^{i+1} M_{i+1}(n-1)]_d \neq \emptyset \Rightarrow [x_n^i M_i(n-1)]_d = x_n^i k[x_1, \ldots, x_{n-1}]_{d-i}$$

equivalently

$$M_{i+1}(n-1)_{d-i-1} \neq \emptyset \Rightarrow M_i(n-1)_{d-i} = k[x_1, \ldots, x_{n-1}]_{d-i}$$

Since this statement is non-vacuous only for $d \geq i+1$ it is more convenient to rewrite it as

$$(*) \qquad M_{i+1}(n-1)_d \neq \emptyset \Rightarrow M_i(n-1)_{d+1} = k[x_1, \ldots, x_{n-1}]_{d+1}$$

for all $i \geq 0$ and $d \geq 0$.

(7) Conversely if $M_0(n-1), M_1(n-1), \ldots, M_r(n-1)$ is a collection of rev-lex segment order ideals in $k[x_1, \ldots, x_{n-1}]$ satisfying the property $(*)$ then

$$M(n) = M_0(n-1) \cup x_n M_1(n-1) \cup x_n^2 M_2(n-1) \cup \cdots \cup x_n^r M_r(n-1)$$

is a rev-lex segment order ideal in $k[x_1, \ldots, x_n]$.

Proof: To see that $M(n)$ is an order ideal it is helpful to arrange the elements of $M(n)$ in the following array. The rows correspond to degree (starting with degree 0 in the top row). The columns starting at the left are to be multiplied by $1, x_n, x_n^2, \ldots$ respectively. For convenience of notation these powers of x_n are left understood. There are $r+1$ columns. Entries in the array can be empty (except for the $\{1\}$'s).

$$\begin{array}{llll}
\{1\} & & & \\
M_0(n-1)_1 & \{1\} & & \\
M_0(n-1)_2 & M_1(n-1)_1 & \{1\} & \\
M_0(n-1)_3 & M_1(n-1)_2 & M_2(n-1)_1 & \{1\} \\
 & \vdots & &
\end{array}$$

If $x \in M(n)$, and $y \neq x$ is a monomial of $k[x_1, \ldots, x_n]$ dividing x then y must be of lower degree than x and contain the same or lower power of x_n. Hence y either occurs directly above x, or to the upper left of x in the array. In the first case $y \in M(n)$ because the $M_i(n-1)$ are order ideals. In the second case $y \in M(n)$ by the two following remarks: (i) The assumption that the $M_i(n-1)$ are order ideals implies that if a particular entry $M_i(n-1)_d$ is everything (i.e. if $M_i(n-1)_d$ is all of $k[x_1, \ldots, x_{n-1}]_d$) then so is any entry above it. (ii) The condition $(*)$ says that if a particular entry in the array is non-empty, then the entry to the left is everything.

This shows that $M(n)$ is an order ideal. That $M(n)$ is a rev-lex segment order ideal follows from $(*)$ and the assumption that the $M_i(n-1)$ are rev-lex segment order ideals.

(8) Given a finite rev-lex segment order ideal $M(n)$ decomposed as

$$M(n) = M_0(n-1) \cup x_n M_1(n-1) \cup x_n^2 M_2(n-1) \cup \cdots \cup x_n^r M_r(n-1)$$

(from section 6) we define $\alpha(M(n)) = r + 1$. Then we have $x_n^r \in M(n)$ since $1 \in M_r(n-1)$. By the definition of rev-lex order $M(n)$ contains all monomials of degree r (or less) in $k[x_1, \ldots, x_n]$. But $x_n^{r+1} \notin M(n)$ since $M_{r+1}(n-1) = \emptyset$. Thus $\alpha(M(n))$ is the smallest degree such that $M(n)$ does not contain all monomials of that degree in $x_1, \ldots, x_n$ (or equivalently, the smallest degree in which the corresponding ideal spanned by $\mathscr{M} \backslash M(n)$ is non-zero).

(9) Define $\sigma(M(n))$ to be the smallest degree such that $M(n)$ is empty in that degree. If there is any monomial of degree d in $M(n)$, $(n > 1)$ then there is a monomial of degree d that does not contain x_n (e.g. x_1^d). Therefore $\sigma(M(n)) = \sigma(M_0(n-1))$ $(n > 1)$.

(10) Recall the condition (*)

$$M_{i+1}(n-1)_d \neq \emptyset \Rightarrow M_i(n-1)_{d+1} = k[x_1, \ldots, x_{n-1}]_{d+1}.$$

This is equivalent to its contrapositive

$$M_i(n-1)_{d+1} \subsetneq k[x_1, \ldots, x_{n-1}]_{d+1} \Rightarrow M_{i+1}(n-1)_d = \emptyset$$

which in turn is equivalent to

$$\sigma(M_{i+1}(n-1)) < \alpha(M_i(n-1)), \; i \geq 0.$$

Sections (6), (7), (10) now give us a recursive construction of rev-lex segment order ideals:

If we decompose a set of monomials $M(n)$ of $k[x_1, \dots, x_n]$ set theoretically as

$$M(n) = M_0(n-1) \cup x_n M_1(n-1) \cup x_n^2 M_2(n-1) \cup \cdots \cup x_n^r M_r(n-1)$$

where the $M_i(n-1)$ are (possibly empty) sets of monomials in $k[x_1, \dots x_{n-1}]$ with $M_r(n-1)$ non-empty then $M(n)$ is a rev-lex segment order ideal in $k[x_1, \dots, x_n]$ if and only if each $M_i(n-1)$ is a rev-lex segment order ideal in $k[x_1, \dots x_{n-1}]$ and

$$\sigma(M_{i+1}(n-1)) < \alpha(M_i(n-1)), \quad \text{for all } i \geq 0.$$

If $n = 1$ then an order ideal in $k[x_1]$ is of the form $\{1, x_1, \dots, x_1^{d-1}\}$ for some $d \geq 1$. Write symbolically $\{d\} = \{1, x_1, \dots, x_1^{d-1}\}$ and define $\sigma(\{d\}) = \alpha(\{d\}) = d$. Also write

$$M(n) = \{M_0(n-1), M_1(n-1), \dots, M_r(n-1)\}$$

where $\sigma(M_{i+1}(n-1)) < \alpha(M_i(n-1))$ for $0 \leq i \leq r-1$ and define $\sigma(M(n)) = \sigma(M_0(n-1)),\ \alpha(M(n)) = r+1$.

(11) Now recall the definition of type vector:

(a) A 1-type vector is a vector of the form $\tau = (d)$ where $d \geq 1$ is a positive integer, together with integers $\alpha(\tau)$ and $\sigma(\tau)$ both defined to be d.

(b) for $n > 1$ an n-type vector $\tau = (\tau_1, \dots, \tau_s)$ is an ordered collection of $(n-1)$-type vectors $\tau_i,\ 1 \leq i \leq s$, where for each $i,\ 1 \leq i \leq s-1$ we have $\sigma(\tau_i) < \alpha(\tau_{i+1})$ together with integers $\alpha(\tau)$ and $\sigma(\tau)$ defined by $\alpha(\tau) = s,\ \sigma(\tau) = \sigma(\tau_s)$.

We now notice that the recursive definition of type vectors is the same as the recursive construction of finite rev-lex segment order ideals (except for the order reversal of the co-ordinates). Thus there is a one-to-one correspondence between finite rev-lex order ideals in $k[x_1, \dots, x_n]$ and n-type vectors.

But by Macaulay, finite rev-lex segment order ideals are in one-to-one correspondence with the Hilbert functions of 0-dimensional graded quotients of $k[x_1, \dots, x_n]$, and the latter are in one-to-one correspondence with Hilbert functions of non-empty finite sets of points in $\mathbb{P}^n$.

Putting all these results together we obtain the desired one-to-one correspondence between n-type vectors and Hilbert functions of non-empty finite sets of points in $\mathbb{P}^n$.

(12) Now I illustrate the above ideas with a couple of examples.

Example 1: Let H be the Hilbert function 1 4 10 13 15 16 17 $\to$, which is the Hilbert function of a set of 17 points in $\mathbb{P}^3$. The Hilbert function of a 0-dimensional quotient of $k[x_1, x_2, x_3]$ corresponding to H is $\Delta H =$ 1 3 6 3 2 1 1 0 $\to$. This corresponds to the order ideal $M(3) =$

$$\begin{array}{llllll}
1 \\
x_1 & x_2 & x_3 \\
x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 \\
x_1^3 & x_1^2x_2 & x_1x_2^2 \\
x_1^4 & x_1^3x_2 \\
x_1^5 \\
x_1^6
\end{array}$$

Here $M_0(2) = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_1^4, x_1^3x_2, x_1^5, x_1^6\}$, which splits up as order ideals $\{1, x_1, x_1^2, x_1^3, x_1^4, x_1^5, x_1^6\}$, $\{1, x_1, x_1^2, x_1^3\}$, $\{1, x_1\}$, and $M_1(2) = \{1, x_1, x_2\}$, which splits up as $\{1, x_1\}, \{1\}$, and $M_2(2) = \{1\}$. Altogether H corresponds to the 3-type vector (((1)),((1),(2)),((2),(4),(7))).

Example 2: Suppose that we start with the 3-type vector (((1),(2)), ((1),(3),(4)),((2),(3),(4),(5),(6))). As an example I will go through the recursive contruction of the rev-lex order ideal corresponding to the 2-type vector ((1),(3),(4)): (4) corresponds to the order ideal $\{1, x_1, x_1^2, x_1^3\}$, (3) corresponds to $\{1, x_1, x_1^2\}$, and (1) corresponds to $\{1\}$. These are combined to form the order ideal $M_1(2) = \{1, x_1, x_1^2, x_1^3, \{1, x_1, x_1^2\}x_2, \{1\}x_2^2\} =$

$$\begin{array}{lll}
1 \\
x_1 & x_2 \\
x_1^2 & x_1x_2 & x_2^2 \\
x_1^3 & x_1^2x_2
\end{array}$$

Similarly ((1),(2)) corresponds to the order ideal $M_2(2) = \{1, x_1, x_2\}$ and ((2),(3), (4),(5),(6)) corresponds to $M_0(2) =$

$$\begin{array}{lllll}
1 \\
x_1 & x_2 \\
x_1^2 & x_1x_2 & x_2^2 \\
x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^4 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \\
x_1^5 & x_1^4x_2 & x_1^3x_2^2 & x_1^2x_2^3 & x_1x_2^4
\end{array}$$

and these three order ideals are combined as $M_0(2) \cup x_3 M_1(2) \cup x_3^2 M_2(2) =$

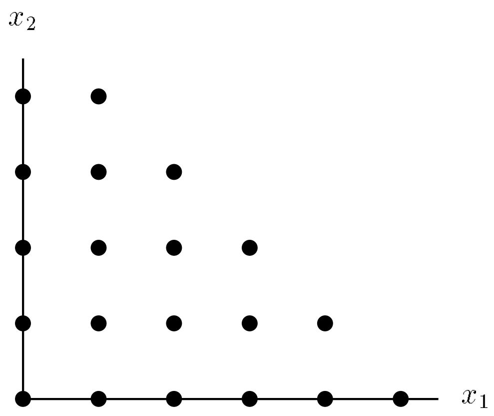
$$\begin{array}{llllllllll}
1 \\
x_1 & x_2 & x_3 \\
x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 \\
x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 & x_1^2x_3 & x_1x_2x_3 & x_2^2x_3 & x_1x_3^2 & x_2x_3^2 \\
x_1^4 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 & x_1^3x_3 & x_1^2x_2x_3 \\
x_1^5 & x_1^4x_2 & x_1^3x_2^2 & x_1^2x_2^3 & x_1x_2^4
\end{array}$$

Now by counting the monomials of each degree we obtain that ΔH = 1 3 6 9 7 5 and $H = 1\ 4\ 10\ 19\ 26\ 31 \rightarrow$ is the Hilbert function corresponding to the type vector (((1),(2)),((1),(3),(4)),((2),(3),(4),(5),(6))).

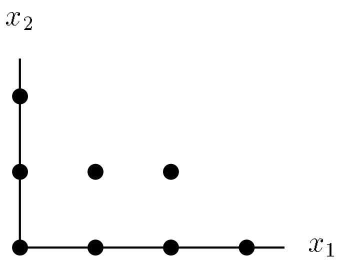
As in [1, section 2], an explicit set of points in $\mathbb{P}^3$ with Hilbert function H can be obtained (in characteristic 0) by taking the exponents of the monomials in the above order ideal (taking $x_0 = 1$ for all points, and listing only the x_1, x_2, x_3 co-ordinates):

(0,0,0), (1,0,0), (2,0,0), (3,0,0), (4,0,0), (5,0,0), (0,1,0), (1,1,0), (2,1,0), (3,1,0), (4,1,0), (0,2,0), (1,2,0), (2,2,0), (3,2,0), (0,3,0), (1,3,0), (2,3,0), (0,4,0), (1,4,0), (0,0,1), (1,0,1), (2,0,1), (3,0,1), (0,1,1), (1,1,1), (2,1,1), (0,2,1), (0,0,2), (1,0,2), (0,1,2)

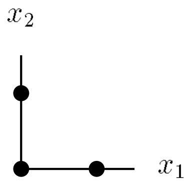
These points can conveniently be graphed in layers according to x_3-value. For $x_3 = 0$ we have



For $x_3 = 1$ we have



and for $x_3 = 2$ we have



These layers can be thought of as corresponding to $M_0(2)$, $M_1(2)$, $M_2(2)$ respectively. In each case α is the number of horizontal rows containing points and σ is the number of points in the bottom row. The condition $\sigma(M_{i+1}(2)) < \alpha(M_i(2))$ has the geometric interpretation that the points of each layer lie strictly within the region where the layer below is completely filled in.

(13) As an application of the above ideas I explain the splitting of a differentiable O-sequence in [3, pages 447-448]. First recall how the splitting works. Let $S = \{b_i\}_{i\geq 0}$ be a differentiable O-sequence. Let $d_1 = b_1 - 1 = n$ and let d_i be the generic value $\binom{n+i-1}{i}$ so long as the sequence $\{c_i\}$ defined by $c_i = b_{i+1} - d_{i+1}$ is non-decreasing, after which we stabilize c_i at its largest value, and define $d_i = b_i - c_{i-1}$. (I know this is sloppy. More precise definitions can be found in [3, page 447].) Let $S_1 = \{c_i\}$ and $S_1' = \{d_i\}$. The claim is then that S_1 and S_1' are both differentiable O-sequences. At the difference sequence level the splitting can be described as follows: start with $\Delta S = \{b_i'\}$ where $b_1' = n$ and let d_i' be the generic value $\binom{n+i-2}{i}$ so long as the difference $c_i' = b_{i+1}' - d_{i+1}'$ is non-negative, after which we define $c_i' = 0$ and $d_i' = b_i'$. Then we have $\Delta S_1' = \{d_i'\}$ and $\Delta S_1 = \{c_i'\}$. Decompose the rev-lex segment ideal corresponding to ΔS as $M_0(n-1) \cup x_n M'(n)$. Then it is clear that

$M_0(n-1)$ is an order ideal in $k[x_1, \ldots, x_{n-1}]$ and $M'(n)$ is an order ideal in $k[x_1, \ldots, x_n]$. Furthermore $\{d'_i\}$ is the Hilbert function of $M_0(n-1)$ and $\{c'_i\}$ is the Hilbert function of $M'(n)$. Therefore $\{d'_i\}$ and $\{c'_i\}$ are O-sequences, and hence S'_1 and S_1 are differentiable O-sequences. This will be illustrated by the following example.

Example. Let $S = \{1\ 4\ 10\ 19\ 30\ 40\ \cdots\}$ (continuing with differences 10) be the Hilbert function of [3, Example 3.6]. Then ΔS is the sequence $\{1\ 3\ 6\ 9\ 11\ 10\ 10 \rightarrow\}$. This is decomposed as follows:

$$\begin{array}{lccccccccccc}
\Delta S & 1 & 3 & 6 & 9 & 11 & 10 & 10 & \rightarrow & & & \\
\{d'_i\} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \rightarrow \\
\hline
\{c'_i\} & & 1 & 3 & 5 & 6 & 4 & 3 & 2 & 1 & 0 & \rightarrow
\end{array}$$

The order ideal corresponding to ΔS is as follows (with $|$ inserted to separate $M_0(2)$ from $x_3M'(3)$).

$$\begin{array}{lllllllllll}
1 & & & & & & & & & & \\
x_1 & x_2 \mid & x_3 & & & & & & & & \\
x_1^2 & x_1x_2 & x_2^2 \mid & x_1x_3 & x_2x_3 & x_3^2 & & & & & \\
x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \mid & x_1^2x_3 & x_1x_2x_3 & x_2^2x_3 & x_1x_3^2 & x_2x_3^2 & & \\
x_1^4 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \mid & x_1^3x_3 & x_1^2x_2x_3 & x_1x_2^2x_3 & x_2^3x_3 & x_1^2x_3^2 & x_1x_2x_3^2 \\
x_1^5 & x_1^4x_2 & x_1^3x_2^2 & x_1^2x_2^3 & x_1x_2^4 & x_2^5 \mid & x_1^4x_3 & x_1^3x_2x_3 & x_1^2x_2^2x_3 & x_1x_2^3x_3 & \\
x_1^6 & x_1^5x_2 & x_1^4x_2^2 & x_1^3x_2^3 & x_1^2x_2^4 & x_1x_2^5 & x_2^6 \mid & x_1^5x_3 & x_1^4x_2x_3 & x_1^3x_2^2x_3 & \\
x_1^7 & x_1^6x_2 & x_1^5x_2^2 & x_1^4x_2^3 & x_1^3x_2^4 & x_1^2x_2^5 & x_1x_2^6 & x_2^7 \mid & x_1^6x_3 & x_1^5x_2x_3 & \\
x_1^8 & x_1^7x_2 & x_1^6x_2^2 & x_1^5x_2^3 & x_1^4x_2^4 & x_1^3x_2^5 & x_1^2x_2^6 & x_1x_2^7 & x_2^8 \mid & x_1^7x_3 & \\
x_1^9 & x_1^8x_2 & x_1^7x_2^2 & x_1^6x_2^3 & x_1^5x_2^4 & x_1^4x_2^5 & x_1^3x_2^6 & x_1^2x_2^7 & x_1x_2^8 & x_2^9 &
\end{array}$$

continuing thereafter with $M_0(2)$ as the first 10 monomials in $k[x_1, x_2]$ and $x_3M'(3)$ empty. We have $M'(3) =$

$$\begin{array}{llllll}
1 & & & & & \\
x_1 & x_2 & x_3 & & & \\
x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & \\
x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 & x_1^2x_3 & x_1x_2x_3 \\
x_1^4 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 & & \\
x_1^5 & x_1^4x_2 & x_1^3x_2^2 & & & \\
x_1^6 & x_1^5x_2 & & & & \\
x_1^7 & & & & &
\end{array}$$

We observe that $\{d_i'\}$ is the Hilbert function of $M_0(2)$ and that $\{c_i'\}$ is the Hilbert function of $M'(3)$. Since $M_0(2)$ and $M'(3)$ are order ideals $\{d_i'\}$ and $\{c_i'\}$ are O-sequences. If we define $S_1' = \{1\ 3\ 6\ 10\ 15\ 21\ 28\ 36\ 45\ 55\ \cdots\}$ (continuing with differences 10) so that $\Delta S_1' = \{d_i'\}$ and $S_1 = \{1\ 4\ 9\ 15\ 19\ 22\ 24\ 25 \rightarrow\}$ so that $\Delta S_1 = \{c_i'\}$ then S_1' and S_1 are differentiable O-sequences and we have the decomposition of [3, section 3].

(14) An initial rev-lex segment of a given degree d is determined by its smallest element. Given $m = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ of degree d denote this rev-lex segment by $L(m)$, the set of all monomials m' of degree d, with $m' \geq m$. Let $\gamma(m)$ be the monomial of degree $d-1$ obtained by lowering by one the exponent of the variable with the smallest index that is actually present in m, and $\delta(m)$ the monomial of degree $d+1$ obtained by increasing by one the exponent of the variable with smallest index that is actually present in m. The following is obvious:

(a) $\gamma\delta(m) = m$ and $\delta\gamma(m) \leq m$

Furthermore

(b) $L(\gamma(m))$ is the initial rev-lex segment in degree $d-1$ consisting of all monomials of degree $d-1$ that divide some element of $L(m)$.

(c) $L(\delta(m))$ is the largest initial rev-lex segment of degree $d+1$, every element of which has all its degree d factors in $L(m)$.

Suppose that we have a set of monomials M of $k[x_1, \dots, x_n]$, which in each degree i is an initial rev-lex segment, determined by a monomial m_i. Then M is an order ideal if and only if equivalently

(d) $\gamma(m_i) \geq m_{i-1}$ equivalently $L(\gamma(m_i)) \subseteq L(m_{i-1})$ for all $i \geq 2$.

(e) $\delta(m_i) \leq m_{i+1}$ equivalently $L(\delta(m_i)) \supseteq L(m_{i+1})$ for all $i \geq 1$.

The equivalence of (d) and (e) is easily checked using (a), and (d) is essentially the definition of an order ideal.

Example: The monomials of degrees 2 and 3 in x_1, x_2, x_3 are:

$$x_1^2\ x_1x_2\ x_2^2\ x_1x_3\ x_2x_3\ x_3^2$$

$$x_1^3\ x_1^2x_2\ x_1x_2^2\ x_2^3\ x_1^2x_3\ x_1x_2x_3\ x_2^2x_3\ x_1x_3^2\ x_2x_3^2\ x_3^3$$

We have $\gamma(x_1^2x_3) = x_1x_3$ and $\delta(x_1x_3) = x_1^2x_3$, whereas $\gamma(x_1x_2x_3) = \gamma(x_2^2x_3) = x_2x_3$ and $\delta(x_2x_3) = x_2^2x_3$. Assertions (b), (c), (d), (e) are all illustrated in this example.

The position of a monomial of a particular degree d in (decreasing) rev-lex order is determined by a binomial expansion: write $m = x_{c_1}x_{c_2}\dots x_{c_d}$ where $1 \le c_1 \le c_2 \le \cdots c_d \le n$ and $c_1 = \cdots = c_e < c_{e+1}$. Those monomials of degree d that are greater than or equal to m can be enumerated as follows:

factor	degree	variables	number
1	d	$1,\cdots,c_d-1$	$\binom{d+c_d-2}{d}$
x_{c_d}	$d-1$	$1,\cdots,c_{d-1}-1$	$\binom{d-1-c_{d-1}-2}{d-1}$
$x_{c_d}x_{c_{d-1}}$	$d-2$	$1,\cdots,c_{d-2}-1$	$\binom{d-2-c_{d-2}-2}{d-2}$
		$\vdots$	
$x_{c_d}\cdots x_{c_{e+2}}$	$e+1$	$1,\cdots,c_{e+1}-1$	$\binom{e+1+c_{e+1}-2}{e+1}$
$x_{c_d}\cdots x_{c_{e+1}}$	e	$1,\cdots,c_e$	$\binom{e+c_e-1}{e}$

In each case the factor is multiplied by all monomials of the degree given in the second column in variables given by the third column. The number of such monomials is given in column four, and the sum of column four is the d-binomial expansion of the position of m in the rev-lex order of monomials of degree d. We have $\delta(m) = x_{c_1}m$ so the corresponding table for $\delta(m)$ is the same except that in each row the degree is increased by 1, increasing the numerator and denominator in each term in the binomial expansion. Then from observations (c) or (e) above we obtain Macaulay's characterization $b_{i+1} \le b_i^{<i>}$ of O-sequences. Similar arguments can be found in [4] where they are actually used to prove Macaulay's theorem.

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