MAXIMAL AND COHEN-MACAULAY PROJECTIVE MONOMIAL CURVES

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Abstract. In this paper we continue the investigation of Cohen-Macaulay projective monomial curves begun in [6]. In the process we introduce maximal curves. Cohen-Macaulay curves are maximal, but not conversely. We show that the number of all curves of degree d that are Cohen-Macaulay grows exponentially, but not as fast as the total number of curves, and also that maximal curves of degree d with sufficiently large embedding dimension relative to d are Cohen-Macaulay.

1. Introduction

First we recall the notation from [6]. We let $\mathbb{N}$ denote the natural numbers $\{0, 1, 2, 3, \cdots\}$. All our semigroups are contained in $\mathbb{N}^n$ for some integer $n > 0$, and contain 0. A numerical semigroup is a subsemigroup of $\mathbb{N}$. The cardinality of a set X is denoted by $|X|$. Let $\mathscr{S} = \{a_1, \ldots, a_k\}$ with $a_i \in \mathbb{N}$, $a_1 < \cdots < a_k = d$ and $\gcd(a_1, \ldots, a_k) = 1$. Let $\Gamma_{\mathscr{S}}$ (or simply Γ if there is no ambiguity) be the numerical semigroup generated by $\mathscr{S}$. To $\mathscr{S}$ we associate the semigroup $S_{\mathscr{S}}$ (or simply S) generated by $\{(d, 0), (d - a_1, a_1), (d - a_2, a_2), \cdots, (d - a_{k-1}, a_{k-1}), (0, d)\}$, and the rings $R_{\mathscr{S}} = K[s^d, s^{d-a_1}t^{a_1}, s^{d-a_2}t^{a_2}, \ldots, t^d]$ (or simply R) and $\mathrm{gr}(\mathscr{S})$ (defined below). Here K can be any field. The ring R is graded by assigning degree 1 to its algebra generators. That is, if $s^it^j \in R$ then s^it^j is assigned degree $(i + j)/d$ as an element of R (clearly $i + j$ is a multiple of d). If $(i, j) \in S$ we will similarly say that (i, j) is of degree $(i + j)/d$.

Note that $R_{\mathscr{S}} \cong K[S]$, the semigroup ring of S. The quotient group of $S_{\mathscr{S}}$ will be denoted $G_{\mathscr{S}}$ (or simply G, or G_d if we wish to indicate what d is) and is equal to $\{(x, y) \in \mathbb{Z}^2 | x + y \equiv 0 \mod d\}$. We assume that $\gcd(a_1, \ldots, a_k) = 1$ because, if not, one could divide all the a_i by

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their gcd and obtain an isomorphic semigroup. Furthermore, assuming that $\gcd(a_1, \ldots, a_k) = 1$ ensures that there are elements of Γ in each congruence class mod d. The generating set $\{(d, 0), (d - a_1, a_1), (d - a_2, a_2), \cdots, (d - a_{k-1}, a_{k-1}), (0, d)\}$ is the Hilbert basis $\mathrm{Hilb}(S)$ of S in the language of [1]. The scheme $C_{\mathscr{S}} = \mathrm{Proj}(R_{\mathscr{S}})$ is a projective monomial curve of degree d whose homogeneous coordinate ring is $R_{\mathscr{S}}$. We will informally identify projective monomial curves of degree d with the sets $\mathscr{S}$ as described above. We will say that the curve $C_{\mathscr{S}_1}$ is contained in the curve $C_{\mathscr{S}_2}$ (of the same degree) if, as sets, $\mathscr{S}_1 \subseteq \mathscr{S}_2$ and will refer to k as the length of $\mathscr{S}$. We will also say that $\mathscr{S}$ is Cohen-Macaulay if and only if $R_{\mathscr{S}}$ is Cohen-Macaulay.

Definition 1.1. Let $\mathscr{S} = \{a_1, \cdots, a_k\}$ with $0 < a_1 < \cdots < a_k = d$ as above. Define the dual of $\mathscr{S}$ to be $\hat{\mathscr{S}} = \{d - a_{k-1}, d - a_{k-2}, \cdots d - a_1, d\}$. The semigroup generated by the elements of $\hat{\mathscr{S}}$ will be denoted by $\hat{\Gamma}$.

It is easily seen that $S_{\hat{\mathscr{S}}} \cong S_{\mathscr{S}}$, by interchanging coordinates. Furthermore Γ is the projection of $S_{\mathscr{S}}$ onto the second coordinate and $\hat{\Gamma}$ is the projection of $S_{\mathscr{S}}$ onto the first coordinate.

Definition 1.2.
- (a) For $x \in \Gamma$ let $\mathrm{ord}_{\mathscr{S}}(x)$ be the smallest integer n such that x can be written as the sum of n elements of $\mathscr{S}$ (counting repetition). Similarly define $\mathrm{ord}_{\hat{\mathscr{S}}}(x)$ for $x \in \hat{\Gamma}$. [7]
- (b) Let $S' = \{\sigma \in G_{\mathscr{S}} |$ for some $a, b \in \mathbb{N}, \sigma + a(0, d) \in S$ and $\sigma + b(d, 0) \in S\}$, or equivalently $S' = \{(\alpha, \beta) \in G_{\mathscr{S}} | \alpha \in \hat{\Gamma}, \beta \in \Gamma\}$.
- (c) Let T be the subsemigroup of S generated by $(d, 0)$ and $(0, d)$. The spanning set $\mathcal{B}$ of S over T is $\{\sigma \in S | \sigma$ cannot be written in the form $\sigma = \sigma' + \tau$ with $\tau \in T,\ \sigma' \in S\}$. [3]

If $x \in \Gamma$ then $\mathrm{ord}_{\mathscr{S}}(x)$ is the smallest integer n such that $(nd - x, x) \in S$. The equivalence of the two statements in Definition 1.2(b) is easy given the previous discussion, and was stated in [6, Lemma 2.6]. Note that S' is a semigroup. We have

Theorem 1.3. *The following conditions are equivalent*
- (a) *$R_{\mathscr{S}}$ is Cohen-Macaulay.*
- (b) *$S' = S$.*
- (c) *For each $a \in [1, d - 1]$, the congruence class of $(d - a, a)$ mod d contains one element of $\mathcal{B}$.*

Proof. The equivalence of (a) and (b) is proved in [2]. These conditions are shown to be equivalent to (c) in [6, Theorem 2.1]. We might note also that $(0, 0)$ is the only element of $\mathcal{B}$ in its equivalence class mod d, which is why $a = 0$ or d could be omitted in the statement of (c). $\square$

In Definition 2.3 below we define maximal projective monomial curves. These are curves such that $S'\backslash S$ contains no element of degree 1. From this and Theorem 1.3 it is immediate that every Cohen-Macaulay projective monomial curve is maximal. Because the test (in terms of $S'\backslash S$) is simpler for maximality than for Cohen-Macaulayness we expect that some questions about maximal curves will be easier than the corresponding question about Cohen-Macaulay curves. For example, we prove in Theorem 2.8 that every maximal curve of degree d and length $k < d$ is contained in a maximal curve of length $k + 1$. But we do not know whether every Cohen-Macaulay curve of degree d and length $k < d$ is contained in a Cohen-Macaulay curve of length $k + 1$. Also asymptotic questions about maximality seem easier. For example we proved in [6] that the fraction of all curves of degree d that are Cohen-Macaulay approaches 0 as $d \to \infty$ by showing that the fraction that are maximal approaches 0 (although without using the terminology maximal). In this paper we study more closely the asymptotic properties of maximal and Cohen-Macaulay curves. We show in section 4 that the number of curves of degree d that are Cohen-Macaulay grows exponentially in d, asymptotically at least as $2^{.5d}$, and that the number that are maximal grows at most as $2^{.906d}$, compared with the (asymptotic) number 2^{d-1} of all curves of degree d. We do not know how the fraction of maximal curves that are Cohen-Macaulay behaves as d becomes large. Our numerical experiments in this regard are inconclusive, but suggest that this fraction either goes to 0 very slowly, or has a lim sup that is less than 1 and greater than 0. In Sections 5 and 6 we show that all maximal curves with $k \geq d/2$ are Cohen-Macaulay.

The following additional notation will be used throughout the paper.

Notation 1.4. Let $[a, b]$ denote the set $\{x|x \in \mathbb{Z}, a \leq x \leq b\}$ and $[a, \infty)$ denote $\{x|x \in \mathbb{Z}, x \geq a\}$. If X and Y are sets of integers and $a \in \mathbb{Z}$, then $a \pm X = \{a \pm x|x \in X\}$ and $X + Y = \{x + y|x \in X, y \in Y\}$.

2. Maximal projective monomial curves.

Our approach to studying the semigroups S corresponding to projective monomial curves is to consider all curves of degree d with the same numerical semigroups Γ and $\hat{\Gamma}$. This divides the set of all curves of degree d into equivalence classes.

Theorem 2.1. *Let Γ and $\hat{\Gamma}$ be two numerical semigroups whose quotient group is $\mathbb{Z}$, both containing a positive integer d and generated by non-negative integers less than or equal to d. Let $\mathscr{F}$ be the set of all semigroups S generated by $X = \{(d, 0) \cup (0, d)\} \cup Y$ where Y is a subset* *of* $\{(d-a,a)|1 \le a \le d-1\}$ *such that the projection of* S *onto the second coordinate is* Γ *and the projection onto the first coordinate is* $\hat{\Gamma}$*. Assume that* $\mathscr{F} \neq \emptyset$*. Then* $\mathscr{F}$ *contains a largest element* S_1 *and a smallest element* S_0*. All elements* S *of* $\mathscr{F}$ *have the same* S' *and only the largest element* S_1 *can have Cohen-Macaulay semigroup ring* $K[S_1]$*. Furthermore if* $|\mathrm{Hilb}(S_1)\backslash\mathrm{Hilb}(S_0)| = a$ *then* $|\mathscr{F}|$ *contains* 2^a *elements.*

Proof. Let S_1 be the semigroup generated by $\{(d,0),(0,d)\}$ and $\{(a,b)\}$ such that $1 \le a,b \le d-1,\ a+b=d$ and $a \in \hat{\Gamma}, b \in \Gamma$. Then clearly $S_1 \in \mathscr{F}$ and S_1 is the largest element of $\mathscr{F}$. Let S_0 be the semigroup generated by $\{(d,0),(0,d)\}$ and all $\{(a,b)\}$ such that $a+b=d$, $1 \le a,b \le d-1$ and either a is a minimal generator of $\hat{\Gamma}$ or b is a minimal generator of Γ (or both). Every element $S \in \mathscr{F}$ must contain all the elements so described (otherwise the projections of S would be too small). It is now clear that $S_0 \in \mathscr{F}$ and that every element of $\mathscr{F}$ must contain S_0. By Definition 1.2(b), $S' = S_0' = S_1'$ for all $S \in \mathscr{F}$. Hence we cannot have $S = S'$ unless $S = S_1$. So the only element of $\mathscr{F}$ which can have Cohen-Macaulay semigroup ring is S_1. Integers $x \in \mathrm{Hilb}(S_1)\backslash\mathrm{Hilb}(S_0)$ might or might not be in $S \in \mathscr{F}$, which yields the final conclusion. □

We should note that for a given d not all pairs $\{\Gamma, \hat{\Gamma}\}$ (where $\Gamma, \hat{\Gamma}$ both contain d and are generated by non-negative integers less than or equal to d) can occur because if b is a minimal generator of Γ then $d-b$ might not be an element of $\hat{\Gamma}$, or vice versa. It is convenient sometimes to describe a numerical semigroup Γ by giving its complement Γ^c (in $\mathbb{N}$). Furthermore we can describe S by giving the corresponding list $\mathscr{S}$, as in the Introduction (with $\mathscr{S}_0$ corresponding to S_0 and $\mathscr{S}_1$ corresponding to S_1).

Some examples are as follows.

Example 2.2. (a) If $\{\Gamma, \hat{\Gamma}\} = \{\mathbb{N}, \mathbb{N}\}$ then $\mathscr{S}_0 = \{1, d-1, d\}$ and $\mathscr{S}_1 = \{1,2,3,\ldots,d-1,d\}$. In this case $|\mathrm{Hilb}(S_1)\backslash\mathrm{Hilb}(S_0)|$ contains $d-3$ elements so $\mathscr{F}$ contains 2^{d-3} elements.

(b) If $d=7$ and $\mathscr{S} = \{2,5,7\}$ then $\Gamma^c = \hat{\Gamma}^c = \{1,3\}$ and $\mathscr{F} = \{\mathscr{S}\}$. The semigroup ring $K[S]$ of $\mathscr{S}$ is not Cohen-Macaulay (because $S'\backslash S = \{(8,6),(6,8),(8,13),(13,8)\}$, all of degree greater than or equal to 2).

(c) If $d=6$ then $\{\Gamma^c, \hat{\Gamma}^c\} = \{\{1\},\{1,2\}\}$ cannot occur. For Γ has minimal generators $\{2,3\}$ and $\hat{\Gamma}$ has minimal generators $\{3,4,5\}$. But then (since 5 is a minimal generator of $\hat{\Gamma}$) we must have $(5,1) \in S$ which is a contradiction since $1 \notin \Gamma$. But if $d \ge 7$ then the pair $\{\{1\},\{1,2\}\}$ does occur, with $\mathscr{S}_0 =$

$\{2, 3, d-5, d-4, d-3, d\}$ and $\mathscr{S}_1 = \{2, 3, \ldots, d-3, d\}$. In fact, any pair $\{\Gamma, \hat{\Gamma}\}$ occurs for sufficiently large d.

Definition 2.3. The curves S_1 will be called maximal.

The sets $\mathscr{F}$ can be very large as shown by Example 2.2(a). Since only the largest element S_1 of $\mathscr{F}$ can be Cohen-Macaulay this suggests that as d goes to infinity the fraction of the curves that are Cohen-Macaulay should approach 0. This indeed was proved to be the case in [6], but by a different approach, without using the sets $\mathscr{F}$.

We now show that each maximal curve is contained in a maximal curve of length one greater, first proving a series of small lemmas. From the definition of S_1 in the proof of Theorem 2.1 and [6, Lemma 2.6] we have

Lemma 2.4. *The curve $\mathscr{S}$ of degree d is maximal if and only if*

$$\mathscr{S} = \{a \in [1, d] | a \in \Gamma, d - a \in \hat{\Gamma}\}$$

if and only if $S' \backslash S$ contains no elements of degree 1.

Note that for all curves, $\Gamma \cap (d - \hat{\Gamma}) = \{a \in [0, d] | a \in \Gamma, d - a \in \hat{\Gamma}\}$. Lemma 2.4 is symmetric in $\mathscr{S}$ and $\hat{\mathscr{S}}$ so $\mathscr{S}$ is maximal if and only if $\hat{\mathscr{S}}$ is maximal. The following observation follows immediately from Lemma 2.4.

Corollary 2.5. *If $\mathscr{S} = \Gamma \cap [1, d]$ then $\mathscr{S}$ is maximal. If $\hat{\Gamma} = \mathbb{N}$ then $\mathscr{S}$ is maximal if and only if $\mathscr{S} = \Gamma \cap [1, d]$.*

Lemma 2.6. *If $\mathscr{S} \subsetneq [1, d] = \{i \in \mathbb{N} | 1 \leq i \leq d\}$ and $\Gamma \cap [1, d] = \mathscr{S}$ then $\hat{\mathscr{S}}$ contains an element smaller than the smallest element of $\mathscr{S}$.*

Proof. Let a be the smallest element of $\mathscr{S}$. The hypotheses imply that $a > 1$. If $a \nmid d$ then $a\lfloor d/a \rfloor \in \Gamma$ (hence in $\mathscr{S}$), $1 \leq d - a\lfloor d/a \rfloor < a$ and $d - a\lfloor d/a \rfloor \in \hat{\mathscr{S}}$ as required. If $a | d$ then there is some element $b \in \mathscr{S}$ such that $a \nmid b$ and $b + a\lfloor (d-b)/a \rfloor \in \Gamma$ (hence in $\mathscr{S}$) with $1 \leq d - b - a\lfloor (d-b)/a \rfloor < a$. Again we obtain an element $d - b - a\lfloor (d-b)/a \rfloor \in \hat{\mathscr{S}}$ that is smaller than a. $\square$

Lemma 2.7. *If $\Gamma \cap [1, d] = \mathscr{S}$ and $\hat{\Gamma} \cap [1, d] = \hat{\mathscr{S}}$ then $\mathscr{S} = [1, d]$.*

Proof. If $\mathscr{S} \subsetneq [1, d]$ then also $\hat{\mathscr{S}} \subsetneq [1, d]$. Suppose that $\mathscr{S} \subsetneq [1, d]$ and that a is the smallest element of $\mathscr{S}$. Necessarily $a > 1$. By Lemma 2.6, $\hat{\mathscr{S}}$ contains an element $\hat{a}$ smaller than a, and by Lemma 2.6 applied to $\hat{\mathscr{S}}$, $\mathscr{S}$ contains an element smaller than $\hat{a}$, hence smaller than a, which is a contradiction. $\square$

Theorem 2.8. *Let $\mathscr{S}$ be any maximal curve of degree d and length $k < d$. Then $\mathscr{S}$ is contained in a maximal curve of length $k+1$ and degree d.*

Proof. Let $\mathscr{S}$ be a maximal curve. By Lemma 2.7 (and the fact that $\mathscr{S}$ is maximal if and only if $\hat{\mathscr{S}}$ is maximal) we may suppose that $\mathscr{S} \subsetneq \Gamma \cap [1,d]$. Let a be the smallest element of $(\Gamma \cap [1,d]) \backslash \mathscr{S}$. We claim that $\mathscr{S}' = \mathscr{S} \cup \{a\}$ is maximal. Let Γ' and $\hat{\Gamma}'$ be the numerical semigroups corresponding to $\mathscr{S}'$. By construction $\Gamma' = \Gamma$, and since $\mathscr{S}$ is maximal we must have $\hat{\Gamma} \subsetneq \hat{\Gamma}'$. By Lemma 2.4, in order to show maximality of $\mathscr{S}'$, it suffices to prove that if $b > a,\ b \in (\Gamma \cap [1,d]) \backslash \mathscr{S}$, then $d-b \notin \hat{\Gamma}'$. But $d-b \notin \hat{\Gamma}$ because $\mathscr{S}$ is maximal and $d-b \notin \hat{\Gamma}'$ because $d-b < d-a$ and $\hat{\Gamma}'$ is generated by $\hat{\Gamma}$ and $d-a$. $\square$

Example 2.9. Here are some examples of the application of Theorem 2.8.

(a) Let $\mathscr{S} = \{4,5,8\}$. This is maximal because $\mathscr{S} = \Gamma \cap [1,8]$. We have $\hat{\mathscr{S}} = \{3,4,8\}$ and $\hat{\Gamma} \cap [1,8] = \{3,4,6,7,8\}$ so add 6 to $\mathscr{S}'$ and hence 2 to $\mathscr{S}$, yielding $\{2,4,5,8\}$ which is maximal.

(b) Let $\mathscr{S} = \{2,4,5,8\}$. Here $\Gamma \cap [1,8] = \{2,4,5,6,7,8\}$ so add 6 yielding $\{2,4,5,6,8\}$ which is maximal. All three of the examples $\{4,5,8\}$, $\{2,4,5,8\}$ and $\{2,4,5,6,8\}$ are Cohen-Macaulay as well as maximal.

(c) The curve $\{2,5,7\}$ is maximal but not Cohen-Macaulay. Applying Theorem 2.8 yields $\{2,4,5,7\}$ which is Cohen-Macaulay. The curve $\{4,5,9\}$ is Cohen-Macaulay. Applying Theorem 2.8 yields $\{4,5,8,9\}$ which is (of course maximal but) not Cohen-Macaulay.

In all the above cases there are several other maximal curves of length $k+1$ containing the given curve. For example $\{3,4,5,8\}$ or $\{1,2,4,5,8\}$. But often there is only one. The curve $\{4,5,9\}$ is the smallest example of a Cohen-Macaulay example on which the construction of Theorem 2.8 yields a non Cohen-Macaulay curve.

Theorem 2.10. *Let $\mathscr{S}$ be a curve of degree d and suppose $a \in [1, d-1]$. Then there are no elements of $S' \backslash S$ (equivalently only one element of $\mathcal{B}$) in the equivalence class of $(d-a, a)$ mod d if and only if the following condition is satisfied (in which case the unique element of $\mathcal{B}$ congruent to $(d-a, a)$ is $(d-a+sd, a+rd)$):*
Let $r \geq 0$ be the smallest integer such that $a+rd \in \Gamma$ and let $s \geq 0$ be the smallest integer such that $d-a+sd \in \hat{\Gamma}$. Then $(d-a+sd, a+rd) \in S$.

Proof. First remark that $r,\ s \in \mathbb{N}$ exist because $\mathscr{S}$ has gcd 1. Recall that $S' = \{(\alpha, \beta) \in G_{\mathscr{S}} | \alpha \in \hat{\Gamma},\ \beta \in \Gamma\}$. The element $(d-a+sd, a+rd)$ is thus the smallest element of S' in the congruence class of $(d-a, a)$ mod d. Furthermore the set of all elements of S' in the congruence class of $(d-a, a)$ mod d is $\{(d-a+id, a+jd) | i \geq s, j \geq r\}$. Thus all elements of S' in the congruence class of $(d-a, a)$ lie in S if and only if $(d-a+sd, a+rd) \in S$. □

Some elementary but useful reformulations of Theorem 2.10 are given by

Corollary 2.11. *In the notation of Theorem 2.10 we always have* $\operatorname{ord}_{\mathscr{S}}(a+rd) \geq r+s+1$, $\operatorname{ord}_{\hat{\mathscr{S}}}(d-a+sd) \geq r+s+1$, *and the following are equivalent:*

(a) $(d-a+sd, a+rd) \in S$
(b) $\operatorname{ord}_{\mathscr{S}}(a+rd) = r+s+1$
(c) $a+jd \notin \Gamma$ *for* $j < \operatorname{ord}_{\hat{\mathscr{S}}}(d-a+sd) - s - 1$
(d) $\operatorname{ord}_{\hat{\mathscr{S}}}(d-a+sd) = r+s+1$
(e) $d-a+jd \notin \hat{\Gamma}$ *for* $j < \operatorname{ord}_{\mathscr{S}}(a+rd) - r - 1$

Proof. We have $(\operatorname{ord}_{\mathscr{S}}(a+rd)d-a-rd, a+rd) \in S$ so $\operatorname{ord}_{\mathscr{S}}(a+rd)d-a-rd \in \hat{\Gamma}$. But $\operatorname{ord}_{\mathscr{S}}(a+rd)d-a-rd = d-a+(\operatorname{ord}_{\mathscr{S}}(a+rd)-r-1)d$ so $\operatorname{ord}_{\mathscr{S}}(a+rd)-r-1 \geq s$ (by the definition of s) or equivalently $\operatorname{ord}_{\mathscr{S}}(a+rd) \geq r+s+1$. Similarly $\operatorname{ord}_{\hat{\mathscr{S}}}(d-a+sd) \geq r+s+1$. If $(d-a+sd, a+rd) \in S$ then $\operatorname{ord}_{\mathscr{S}}(a+rd) \leq (d-a+sd+a+rd)/d = r+s+1$ (no matter what r and s mean) so altogether (a) implies (b). If (b) holds then $((r+s+1)d-(a+rd), a+rd) = (d-a+sd, a+rd) \in S$ so (b) implies (a). Similarly (a) and (d) are equivalent. If $\operatorname{ord}_{\mathscr{S}}(a+rd) = r+s+1$ then $\operatorname{ord}_{\mathscr{S}}(a+rd)-r-1 = s$ so (e) holds by the definition of s. Thus (b) implies (e). Similarly (d) implies (c). If (c) holds then $\operatorname{ord}_{\hat{\mathscr{S}}}(d-a+sd)-s-1 \leq r$ by the definition of r so $\operatorname{ord}_{\hat{\mathscr{S}}}(d-a+sd) \leq r+s+1$ which, together with the initial observation, implies (d). Similarly (e) implies (b), completing the proof. □

A curve $\mathscr{S}$ of degree d is Cohen-Macaulay if and only if the conditions of Theorem 2.10 or Corollary 2.11 hold for all $a \in [1, d-1]$. Furthermore $\mathscr{S}$ is maximal if and only if these conditions hold whenever $r = 0$ and $s = 0$. We have found the following specializations of Theorem 2.10 particularly useful.

Corollary 2.12. *With the notation of Theorem 2.10, suppose that* $\mathscr{S}$ *is maximal and* $a \in \Gamma$ *(i.e.* $r = 0$*) with* $\operatorname{ord}_{\mathscr{S}}(a) = 2$*. Then Corollary 2.11 holds. Dually, if* $d-a \in \hat{\Gamma}$ *(i.e.* $s = 0$*) with* $\operatorname{ord}_{\hat{\mathscr{S}}}(d-a) = 2$*, then Corollary 2.11 also holds.*

Proof. For the $r = 0$ case we need only check condition 2.11(e) for $j = 0$, where it holds by the maximality of $\mathscr{S}$. (Consequently $s = \mathrm{ord}_{\mathscr{S}}(a) - 1$.) The dual $s = 0$ statement is proved similarly. □

Corollary 2.13. *Let $\mathscr{S}$ be a curve of degree d. Suppose that*

(a) *For all $a \in [1, d]$ either $a \in \hat{\Gamma}$ or $d - a \in \Gamma$ (or possibly both),*
(b) $\mathrm{ord}_{\mathscr{S}}(a) \leq 2$ *for all* $a \in [1, d] \cap \Gamma$ *and* $\mathrm{ord}_{\hat{\mathscr{S}}}(a) \leq 2$ *for all* $a \in [1, d] \cap \hat{\Gamma}$.

Then $\mathscr{S}$ is Cohen-Macaulay if and only if $\mathscr{S}$ is maximal.

Proof. Cohen-Macaulay always implies maximal. If $\mathscr{S}$ is maximal the present Corollary follows by applying Corollary 2.12 to each point $(d-a, a)$, $a \in [1, d-1]$. □

Remark 2.14. (1) Corollary 2.13 is crucial to our later results. It could of course have been given a direct proof, without going through the somewhat complicated looking Theorem 2.10 and Lemma 2.11. But many curves are Cohen-Macaulay without satisfying the (apparently fairly strong) hypotheses of Corollary 2.13. Therefore it was of interest to us to give a *characterization* of Cohen-Macaulay that included Lemma 2.13.
(2) The curve $\{4, 5, 9\}$ is Cohen-Macaulay and satisfies hypothesis (b) of Corollary 2.13. But $2 \notin \hat{\Gamma}, 7 \notin \Gamma$ so hypothesis (a) is not satisfied. The curve $\{4, 5, 6\}$ is Cohen-Macaulay and satisfies hypothesis (a) of 2.13 but hypothesis (b) does not hold because $\mathrm{ord}_{\hat{\mathscr{S}}}(5) = 3$.
(3) If a curve satisfies (a) or (b) of Corollary 2.13, then so does any larger curve of degree d.

Theorem 2.15. *Suppose that $\mathscr{S}$ is a curve of degree d such that Γ and $\hat{\Gamma}$ both contain all integers greater than or equal to d. Suppose furthermore that $\mathscr{S}$ is Cohen-Macaulay. Then* $\mathrm{ord}_{\mathscr{S}}(a) \leq 2$ *for all* $a \in [1, d] \cap \Gamma$ *and* $\mathrm{ord}_{\hat{\mathscr{S}}}(a) \leq 2$ *for all* $a \in [1, d] \cap \hat{\Gamma}$.

Proof. Suppose (for example) that there exists $\alpha \in \hat{\Gamma} \cap [1, d]$ with $\mathrm{ord}_{\hat{\mathscr{S}}}(\alpha) > 2$. Then $(\alpha, 2d - \alpha) \notin S$. But by assumption $2d - \alpha \in \Gamma$. Thus $(\alpha, 2d - \alpha) \in S' \backslash S$ and hence $\mathscr{S}$ is not Cohen-Macaulay. □

Remark 2.16. (1) Hypothesis (a) of Corollary 2.13 is symmetric in Γ and $\hat{\Gamma}$.
(2) If Γ and $\hat{\Gamma}$ are both fixed then it is clear that for sufficiently large d there is a maximal curve corresponding to Γ and $\hat{\Gamma}$, and this curve satisfies the hypotheses of Corollary 2.13 (hence is Cohen-Macaulay).

(3) Theorem 2.15 gives a quick proof that $\mathscr{S} = \{2,5,7\}$ is not Cohen-Macaulay. For $6 \in [1,7] \cap \Gamma$ and $\mathrm{ord}_{\mathscr{S}}(6) = 3$.
(4) The first hypothesis of Theorem 2.15 is satisfied most of the time, as is shown by the following lemma.

Lemma 2.17. *Let c_d denote the number of projective monomial curves of degree d and m_d denote the number for which Γ contains all integers greater than or equal to d. Then $\lim_{d\to\infty} m_d/c_d = 1$.*

Proof. As usual we identify projective monomial curves of degree d with sets $\mathscr{S} = \{a_1, \ldots, a_k\}$ such that $0 < a_1 < \ldots a_k = d$ and $\gcd(a_1 \ldots a_k) = 1$. We saw in [6] that $\lim_{d\to\infty} c_d/2^{d-1} = 1$, so it will suffice to show that $\lim_{d\to\infty} m_d/2^{d-1} = 1$. Let $\mathscr{S} = \{a_1, \ldots, a_k\}$ be a projective monomial curve of degree d. If $a_1 = 1$, then $\Gamma = \mathbb{N}$ and there are 2^{d-2} such $\mathscr{S}$. If $a_1 = a > 1$ and $at+1 \in \mathscr{S}$, the semigroup generated by a and $at+1$ alone will contain all integers greater than or equal to $(a-1)at$. As long as $(a-1)at \le d$, the largest integer not in Γ will be less than or equal to d. There are $\lfloor \frac{d}{a(a-1)} \rfloor$ positive integer values of t that satisfy this condition. There are a total of 2^{d-1-a} sets of the form $\{a_1, \ldots, a_k | 0 < a = a_1 < \ldots < a_k = d\}$ with no condition on the gcd. The number of these that contain no element of the form $at+1$ with $(a-1)at \le d$ is $2^{d-1-a-\lfloor d/(a(a-1)) \rfloor}$. Therefore the total number of sets containing at least one such $at+1$ is

$$2^{d-1-a} - 2^{d-1-a-\lfloor \frac{d}{a(a-1)} \rfloor}$$

(Note that all of these sets are guaranteed to have $\gcd(a_1, \ldots, a_k) = 1$ and hence are curves.) We therefore have

$$m_d \ge 2^{d-2} + \sum_{\substack{a \ge 2 \\ a(a-1) \le d}} 2^{d-1-a} - 2^{d-1-a-\lfloor \frac{d}{a(a-1)} \rfloor}$$

Now

$$\lim_{d\to\infty} (2^{d-2} + \sum_{\substack{a \ge 2 \\ a(a-1) \le d}} 2^{d-1-a})/2^{d-1} = \sum_{a=1}^{\infty} 2^{-a} = 1$$

and

$$\begin{aligned} \sum_{\substack{a \ge 2 \\ a(a-1) \le d}} 2^{d-1-a-\lfloor \frac{d}{a(a-1)} \rfloor}/2^{d-1} &\le \sum_{\substack{a \ge 2 \\ a(a-1) \le d}} 2^{-a-\frac{d}{a(a-1)}+1} \\ &\le \sum_{\substack{a \ge 2 \\ a(a-1) \le d}} 2^{-a-\frac{d}{a^2}+1} \end{aligned}$$

For fixed d the maximum value of $-a-\frac{d}{a^2}+1$ occurs when $a=(2d)^{1/3}$, i.e. $-a-\frac{d}{a^2}+1 \le 1-3(\frac{d}{4})^{1/3}$. Since $(\sqrt{d}+1)\sqrt{d} > d$, the number of terms such that $a \ge 2$ and $a(a-1) \le d$ is less than or equal to $\sqrt{d}$. Therefore

$$\begin{aligned}\lim_{d\to\infty} \sum_{\substack{a\ge 2\\ a(a-1)\le d}} 2^{d-1-a-\lfloor \frac{d}{a(a-1)}\rfloor}/2^{d-1} &\le \lim_{d\to\infty} \sum_{\substack{a\ge 2\\ a(a-1)\le d}} 2^{-a-\frac{d}{a^2}+1} \\ &\le \lim_{d\to\infty} \sqrt{d}2^{1-3(\frac{d}{4})^{1/3}} \\ &= 0\end{aligned}$$

and the result follows. $\square$

Corollary 2.18. *The fraction of curves of degree d such that Γ and $\widehat{\Gamma}$ both contain all integers greater than or equal to d approaches 1 as d goes to infinity.*

Proof. By the previous Lemma, the fraction of curves of degree d such that Γ contains all integers greater than or equal to d approaches 1 as d goes to infinity. Dually, the fraction of curves where $\widehat{\Gamma}$ has this property also approaches 1. If A_d and B_d are sets of curves of degree d and $\lim_{d\to\infty}|A_d|/c_d = \lim_{d\to\infty}|B_d|/c_d = 1$, then $\lim_{d\to\infty}|A_d\cap B_d|/c_d = \lim_{d\to\infty}|A_d|/c_d+\lim_{d\to\infty}|B_d|/c_d-\lim_{d\to\infty}|A_d\cup B_d|/c_d = 1+1-1=1$ and we are done. $\square$

3. Some examples of maximal curves.

In this section we give some examples of maximal curves of degree d. In this regard recall also Corollary 2.5.

Lemma 3.1. *Suppose that (aside from d) $\mathscr{S}$ contains only integers in the interval $\{a|d/2 \le a < d\}$. Then $\mathscr{S}$ is maximal.*

Proof. This is follows from Corollary 2.5, once one notes that the sum of any two elements in $\mathscr{S}$ is greater than or equal to d. $\square$

By duality if (aside from d) $\mathscr{S}$ contains only integers in the interval $\{a|1 \le a \le d/2\}$ then $\mathscr{S}$ is also maximal.

Lemma 3.2. *Suppose that (aside from d) $\mathscr{S}$ contains only integers in the interval $\{a|d/3 \le a \le 2d/3\}$. Then $\mathscr{S}$ is maximal.*

Proof. We use Lemma 2.4 with the inclusion $\mathscr{S} \subseteq \{a \in [1,d] | a \in \Gamma, d-a \in \hat{\Gamma}\}$ true by definition. Conversely suppose that $a \in [1,d]$, $a \in \Gamma$ and $d-a \in \hat{\Gamma}$. By the definition of Γ, every element of Γ is greater than or equal to $d/3$. Also here $\hat{\Gamma}$ is generated by integers greater than or equal to $d/3$. Therefore $a \geq d/3$ and $d-a \geq d/3$. From the last inequality we have $a \leq 2d/3$ so altogether $d/3 \leq a \leq 2d/3$. If $d/3 \leq a < 2d/3$ then a cannot be the sum of two or more elements of $\mathscr{S}$ so $a \in \Gamma$ implies $a \in \mathscr{S}$. If $a = 2d/3$ (supposing that 3 divides d) then the assumption that $d - a = d/3 \in \hat{\Gamma}$ implies that $d/3 \in \hat{\mathscr{S}}$ from which it follows that $a \in \mathscr{S}$. □

In both cases there will be a small fraction of possibilities that are not relatively prime. However it is clear that the fraction of these cases is very small and hence as d goes to infinity the number of maximal curves of degree d grows at least as fast as $2^{d/2}$.

Now we consider the examples of [4]. As in [6], we will replace the d's in [4] by δ to avoid conflict with our use of d to represent degree.

Theorem 3.3. *The examples of* [4] *with* $\delta > 1$ *are all maximal.*

Proof. These examples are of the form $\mathscr{S} = \{\delta, m, m+\delta, ..., m+(p+1)\delta\} = \{\delta, m_0, ..., m_{p+1},\}$ where $m_i = m + i\delta,\ 0 \leq i \leq p+1,\ p \geq 0,\ \delta > 1$ and $\gcd(m,\delta) = 1$. We have that Γ is generated by δ and m, and that $\hat{\mathscr{S}} = \{\delta, 2\delta, ..., (p+1)\delta, m+p\delta, m+(p+1)\delta\}$. If $\delta > m$ then $\hat{\Gamma} \cap [1,d] = \hat{\mathscr{S}}$ so $\hat{\mathscr{S}}$ is maximal by Corollary 2.5 and hence also $\mathscr{S}$ is maximal. If $\delta < m$ then $\hat{\Gamma} \cap [1,d]$ consists of $\hat{\mathscr{S}}$ together with additional integers $i\delta$ so long as $i\delta < m + (p+1)\delta$ and $\{x \in [1,d] | d - x \in \hat{\Gamma}\}$ consists of $\mathscr{S}$ together with the integers $m-\delta, m-2\delta, \dots m-i\delta \dots$ so long as $m - i\delta > 0$. But the latter integers are not in Γ (because we assume that $\delta > 1$) so $\{a \in [1,d] | a \in \Gamma, d-a \in \hat{\Gamma}\} \subseteq \mathscr{S}$. Always $\mathscr{S} \subseteq \{a \in [1,d] | a \in \Gamma, d-a \in \hat{\Gamma}\}$ so $\mathscr{S}$ is maximal by Lemma 2.4 completing the proof. □

If $\delta = 1$ the examples of [4] are not maximal (except in the trivial case $\mathscr{S} = [1,d]$). If $\delta = 1$ the above proof fails because in this case $\Gamma = \mathbb{N}$ and $m - \delta \in \Gamma$.

If $\mathscr{S} = \{\delta, m, m+\delta, ..., m+(p+1)\delta\}$ then it was proved in [4] that $\mathscr{S}$ is Cohen-Macaulay if and only if $\delta = m-1$ or $m < \delta < m+p+1$. We can start with a given degree d and choose δ small compared to d. Take $\mathscr{S}$ to consist of d, δ and the integers $d-\delta, d-2\delta, \dots, d-(p+1)\delta$ with p not too large. Then $m = d-(p+1)\delta$ and the resulting $\mathscr{S}$ will not be Cohen-Macaulay so long as $d-(p+1)\delta - 1 > \delta$ or equivalently $d > (p+2)\delta + 1$. The number of such choices for pairs $\{p, \delta\}$ with $p \geq 0, \delta \geq 1$ satisfying this inequality grows at least linearly in d so there are lots of maximal curves which are not Cohen-Macaulay. Note that for given d and δ there will be at most one value of p so that

$1 \le d-(p+1)\delta < \delta$ so we cannot so easily generate maximal non-Cohen-Macaulay examples using the criterion $\delta \ge m+p+1$. We do not know if the number of maximal non-Cohen-Macaulay curves of a given degree d grows exponentially in d or not.

Example 3.4. The infinite family of [5] is maximal but not Cohen-Macaulay. In this paper it was proved that the curves $\mathscr{S} = \{-1+6x, 3+6x, 5+6x, 8+12x\}$ are not Cohen-Macaulay for x an integer greater than or equal to 1, and have h-vector with no negative values. Here we observe that all these curves are maximal. If $x=1$ then $\Gamma \cap [1,d] = \{5,9,10,11,14,15,16,18,19,20\}$ and $(20-\hat{\Gamma}) \cap [1,20] = \{2,5,9,11,20\}$ so $\Gamma \cap (20-\hat{\Gamma}) = \{5,9,11,20\} = \mathscr{S}$ so $\mathscr{S}$ is maximal by Lemma 2.4. If $x>1$ then $\Gamma \cap [1,d]$ contains $\{-1+6x, 3+6x, 5+6x, 8+12x\}$ and some integers in the range $[5+6x, 8+12x]$. Similarly $(d-\hat{\Gamma}) \cap [1,d] = \{2, -1+6x, 3+6x, 5+6x, 8+12x\}$ so $\Gamma \cap (d-\hat{\Gamma}) = \mathscr{S}$ and again $\mathscr{S}$ is maximal by Theorem 2.4. In the investigation leading to the paper [5], many other examples of curves with $k=4$ which are not Cohen-Macaulay were found. These were not classified, but we checked many of these, and did not find any that were not maximal.

4. Asymptotics

In this section we show that the fraction of curves of degree d that are Cohen-Macaulay grows exponentially. First we introduce some notation.

Definition 4.1. Let c_d be the number of curves of degree d, cm_d the number of curves of degree d that are Cohen-Macaulay, and ncm_d the number that are not.

We show that cm_d grows asymptotically at least as $2^{d/2}$, compared with c_d which grows as 2^d. More precise statements will follow.

We start with a simple $\mathscr{S}$, consisting of a narrow interval I of consecutive integers in the centre of $[0,d]$ together with d and the interval $J = (I+I) \cap [1, d-1]$ which we assume includes $d-1$. Let $\overline{J} = J \cup \{d\}$. Then $\mathscr{S} = \Gamma \cap [1,d]$ so $\mathscr{S}$ is maximal by Lemma 2.5. Furthermore $d-1 \in \mathscr{S}$ implies $1 \in \hat{\mathscr{S}}$, so $\hat{\Gamma} = \mathbb{N}$. The dual set $\hat{\mathscr{S}}$ consists of a narrow interval $d-J$ containing 1, an interval $d-I$ in the centre of $[0,d]$ and d. We can add any elements between I and J to $\mathscr{S}$ without affecting the fact that $\mathscr{S} = \Gamma \cap [1,d]$ (so that $\mathscr{S}$ remains maximal by Lemma 2.5) or that $\hat{\Gamma} = \mathbb{N}$. Equivalently we can add any set X of elements between $d-J$ and $d-I$ to $\hat{\mathscr{S}}$ while still keeping $\mathscr{S}$ maximal and $\hat{\Gamma} = \mathbb{N}$. For relatively small X we can arrange that $\operatorname{ord}_{\hat{\mathscr{S}}}(x) \le 2$ for all $x \in [1,d]$.

To see this notice that the following elements of $[0,d]$ are of $\hat{\mathscr{S}}$-order less than or equal to 2: $(d-\overline{J})+(d-\overline{J})$ which is a narrow interval of consecutive integers containing 0, $(d-\overline{J})+(d-I)$ which is a narrow interval in the centre of $[0,d]$, and $\{d\}\cup(((d-I)+(d-I))\cap[0,d-1])$, which (by choice of I) is a narrow interval of consecutive integers containing d. (It is convenient to leave 0 in $d-\overline{J}$, so that $(d-\overline{J})+(d-\overline{J})$ contains $d-\overline{J}$, and $(d-\overline{J})+(d-I)$ contains $d-I$, the elements of $d-\overline{J}$ and $d-I$ being of $\hat{\mathscr{S}}$-order less than or equal to 1.) If we now choose X so that elements of $X+(d-\overline{J})$ cover the interval between $(d-\overline{J})+(d-\overline{J})$ and $(d-\overline{J})+(d-I)$, and so that elements of $X+(d-I)$ cover the interval between $(d-\overline{J})+(d-I)$ and $(d-I)+(d-I)$, then all elements of $[1,d]$ will have $\hat{\mathscr{S}}$-order less than or equal to 2. Since we still have $\mathscr{S}=\Gamma\cap[1,d]$, all elements of $[1,d]$ have $\mathscr{S}$-order one. Therefore by Corollary 2.13, $\mathscr{S}$ is Cohen-Macaulay. We still have almost $d/2$ elements between I and J that we can add to $\mathscr{S}$, with $\mathscr{S}$ remaining Cohen-Macaulay, again by Corollary 2.13, from which our claim about the growth of cm_d will follow.

We now carry out the above idea explicitly. Let $n\geq 2$ be a fixed integer, $\ell=\lfloor d/(2n+1)\rfloor$ and $I=[n\ell,(n+1)\ell]$. In order for the above procedure to work we require $2(n+1)\ell\geq d-1$. A sharp bound for this is $d\geq(2n)^2-1$ (in the sense that if $d=(2n)^2-2$ then $2(n+1)\ell<d-1$ and if $d\geq(2n)^2-1$ then $2(n+1)\ell\geq d-1$). So assume that $d\geq(2n)^2-1$. Then $\mathscr{S}$ consists of the intervals $[n\ell,(n+1)\ell]$, $[2n\ell,d]$ and $\hat{\mathscr{S}}$ consists of the intervals $J_1=[1,d-2n\ell], J_2=[d-(n+1)\ell,d-n\ell]$ together with d. The interval J_1 contains $d-2n\ell\geq\ell$ elements, the interval J_2 contains $\ell+1$ elements, with $(n-1)\ell$ integers between them, namely the interval $[d-2n\ell+1,d-(n+1)\ell-1]$ (and it is the latter interval from which we may choose additional elements for $\hat{\mathscr{S}}$) . There are $n\ell-1$ integers in the interval between J_2 and d. The elements of $\hat{\mathscr{S}}$ already of $\hat{\mathscr{S}}$-order less than or equal to 2 are $I_1=[1,2d-4n\ell]$, $I_2=[d-(n+1)\ell,2d-3n\ell]$, and $I_3=[2d-2(n+1)\ell,d]$. (These are the intervals denoted $(d-\overline{J})+(d-\overline{J})$, $(d-\overline{J})+(d-I)$, and $(d-I)+(d-I)$, in the previous paragraph, except that we have removed 0 from the first, and any elements greater than d from the third. Note that $2d-2n\ell>d$ so in fact $(d-I)+(d-I)$ contains d.) Between I_1 and I_2 there are $d-(n+1)\ell-(2d-4n\ell)-1=(3n-1)\ell-d-1\leq(n-2)\ell-1$ integers and between I_2 and I_3 there are $2d-2(n+1)\ell-(2d-3n\ell)-1=(n-2)\ell-1$ integers. We can cover the interval between I_2 and I_3 with at most $n-2$ translations f_i+J_2 of J_2 by adding the integers $f_i=d-(2n-1)\ell+1+i(\ell+1),\ 0\leq i\leq n-3$ to $\hat{\mathscr{S}}$. (If $n=2$, I_1

and I_2 already overlap, as do I_2 and I_3, so all elements of $[1, d]$ are of $\hat{\mathscr{S}}$-order less than or equal to 2, and no extra elements are needed.) Similarly the interval between I_1 and I_2 can be covered with at most $n-2$ translations $g_i + \overline{J}_1$ of $\overline{J}_1 = J_1 \cup \{0\}$ by adding to $\hat{\mathscr{S}}$ the elements $g_i = 2d - 4n\ell + 1 + i(\ell+1),\ 0 \le i \le n-3$. Omit any of the f_i or g_i that are greater than or equal to the smallest element $d - (n+1)\ell$ of J_2. The interval between J_1 and J_2 contains $(n-1)\ell$ elements and we have added at most $2n-4$ of these to $\hat{\mathscr{S}}$ so there remain at least $(n-1)\ell - 2n + 4$ elements that we are free to choose or not, obtaining in each case a Cohen-Macaulay curve. This yields

Theorem 4.2. *For a fixed $n \ge 2$ and $d \ge (2n)^2 - 1$,*

$$cm_d > 2^{(n-1)\lfloor \frac{d}{2n+1} \rfloor - 2n + 4}.$$

Corollary 4.3. $\liminf_{d\to\infty} cm_d^{1/d} \ge 2^{1/2} \approx 1.41421$.

Finally, we find an upper bound.

Theorem 4.4. $\limsup_{d\to\infty} cm_d^{1/d} \le 2(\sqrt{3}/2)^{\gamma} \approx 1.87384 \approx 2^{.906}$, *where* $\gamma = \frac{\ln 2}{2\ln 2 - \ln(\sqrt{3}/2)}$.

Proof. As in the proof of [6, Theorem 5.3] let $C'_d = \{\{a_1, a_2, \ldots, a_k\} | a_i \in \mathbb{N}, 0 < a_1 < a_2 < \ldots < a_k = d\}$, $CM'_d = \{\mathscr{S} \in C'_d | R_{\mathscr{S}}$ is Cohen-Macaulay$\}$, and $NCM'_d = C'_d \setminus CM'_d$ (temporarily dropping the condition that $\gcd(a_1, \ldots, a_k) = 1$). Let $C'_{d,x} = \{\mathscr{S} \in C'_d | x \notin \mathscr{S}\}$, $CM'_{d,x} = CM'_d \cap C'_{d,x}$, and $NCM'_{d,x} = C'_{d,x} \setminus CM'_{d,x}$. Now $NCM'_d = \cup_{0<x<d} NCM'_{d,x}$, so we would like to have a positive lower bound for $|NCM'_{d,x}|$ on some range of x in order to apply [6, Corollary 5.2], which says that if we have a collection of nonempty finite sets $A_i,\ 1 \le i \le n$, and a subset $X_i \subseteq A_i$ for each i, such that $|X_i|/|A_i| \ge \alpha$ for all i then

$$\frac{|\cup X_i|}{|\cup A_i|} \ge \frac{\alpha}{(1-\alpha)n + \alpha}.$$

As in the proof of [6, Theorem 5.3] we have

$$\begin{aligned} \frac{|NCM'_{d,x}|}{|C'_{d,x}|} &\ge \frac{(2^{x-1} - 3^{\lfloor (x-1)/2 \rfloor})(2^{d-x-1} - 3^{\lfloor (d-x-1)/2 \rfloor})}{2^{d-2}} \\ &= \left(1 - \frac{3^{\lfloor (x-1)/2 \rfloor}}{2^{x-1}}\right)\left(1 - \frac{3^{\lfloor (d-x-1)/2 \rfloor}}{2^{d-x-1}}\right) \\ &=: f(x). \end{aligned}$$

Now we replace $1/4$ in the proof of [6, Theorem 5.3] by $\lambda, 0 < \lambda < 1/2$. Let I_λ be the set of all integers in the interval $[\lambda d, (1-\lambda)d]$ and consider $\cup_{x \in I_\lambda} NCM'_{d,x}$. Let $g(t) = (1 - 3^{(t-1)/2}/2^{t-1})(1 - 3^{(d-t-1)/2}/2^{d-t-1})$,

$1 \le t \le d-1$. For $x \in I_\lambda$ we have $|NCM'_{d,x}|/|C'_{d,x}| \ge f(x) \ge g(x) \ge g(\lambda d)$. Note that $g(t)$ is symmetric about $d/2$ and monotone increasing on the interval $[1, d/2]$. The function $f(x)$ is somewhat jagged, equal to $g(x)$ when x is an odd integer, and larger than $g(x)$ at other real values in $[1, d-1]$. Taking $\alpha(d,\lambda) = g(\lambda d)$, noting that there are $n(d,\lambda) = d - 2\lceil \lambda d \rceil + 1$ subsets in our union, and applying [6, Corollary 5.2] we obtain

$$\frac{|\cup_{x\in I_\lambda} NCM'_{d,x}|}{|\cup_{x\in I_\lambda} C'_{d,x}|} \ge \frac{\alpha(d,\lambda)}{(1-\alpha(d,\lambda))n(d,\lambda) + \alpha(d,\lambda)}.$$

Now $\cup_{x\in I_\lambda} C'_{d,x} = \{\mathscr{S} \in C'_d |$ there is an $x \in I_\lambda$ such that $x \notin \mathscr{S}\}$, so its complement in C'_d consists of all $\mathscr{S}$ containing I_λ. Since each such $\mathscr{S}$ is determined by a subset of the $2\lceil \lambda d \rceil - 2$ element set $[1, d-1] \backslash I_\lambda$, there are $2^{2\lceil \lambda d \rceil - 2} = 2^{d-1-n(d,\lambda)}$ elements in the complement. As $|C'_d| = 2^{d-1}$, we have $|\cup_{x\in I_\lambda} C'_{d,x}| = 2^{d-1} - 2^{d-1-n(d,\lambda)}$ and therefore (for all $\lambda,\ 0 < \lambda < d/2$))

$$\begin{aligned}
\frac{|NCM'_d|}{|C'_d|} &\ge \frac{|\cup_{x\in I_\lambda} NCM'_{d,x}|}{|\cup_{x\in I_\lambda} C'_{d,x}|} \frac{|\cup_{x\in I_\lambda} C'_{d,x}|}{|C'_d|} \\
&\ge \frac{\alpha(d,\lambda)}{(1-\alpha(d,\lambda))n(d,\lambda) + \alpha(d,\lambda)} \frac{2^{d-1} - 2^{d-1-n(d,\lambda)}}{2^{d-1}} \\
&= \frac{\alpha(d,\lambda)}{(1-\alpha(d,\lambda))n(d,\lambda) + \alpha(d,\lambda)} \left(1 - 2^{-n(d,\lambda)}\right) \\
&=: F(d,\lambda)
\end{aligned}$$

Our curves are just the elements of C'_d that have relatively prime coordinates. It was shown in the proof of [6, Theorem 5.3] that $c_d = 2^{d-1} - h(d)$ where $0 < h(d) \le d2^{d/2-1}$, so that asymptotically almost all elements of C'_d are relatively prime. Now we have $ncm_d \ge |NCM'_d| - h(d)$ so that

$$\frac{ncm_d}{2^{d-1}} \ge \frac{|NCM'_d|}{2^{d-1}} - \frac{h(d)}{2^{d-1}}$$

and

$$\frac{cm_d}{2^{d-1}} = \frac{c_d}{2^{d-1}} - \frac{ncm_d}{2^{d-1}} = 1 - \frac{h(d)}{2^{d-1}} - \frac{ncm_d}{2^{d-1}} \le 1 - \frac{|NCM'_d|}{2^{d-1}} \le 1 - F(d,\lambda).$$

Then we have

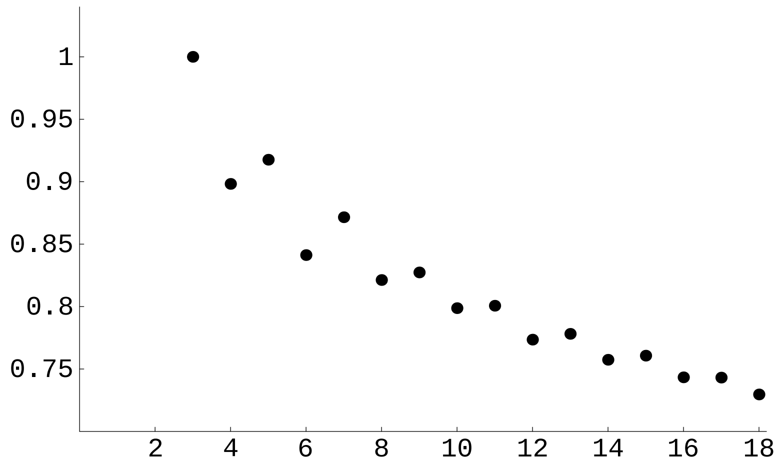
$$1 - F(d,\lambda) = \frac{(1-\alpha(d,\lambda))\frac{n(d,\lambda)}{\alpha(d,\lambda)}}{1 + (1-\alpha(d,\lambda))\frac{n(d,\lambda)}{\alpha(d,\lambda)}} + \frac{2^{-n(d,\lambda)}}{1 + (1-\alpha(d,\lambda))\frac{n(d,\lambda)}{\alpha(d,\lambda)}}$$

Note that $\alpha(d,\lambda) = (1-\beta^{\lambda d-1})(1-\beta^{(1-\lambda)d-1})$ where $\beta = \sqrt{3}/2 = .866025$. Then we have for fixed λ, as $d \to \infty$, $\alpha(d,\lambda) \to 1$, and the dominant term in $1-\alpha(d,\lambda)$ is $\beta^{\lambda d-1}$. Furthermore $n(d,\lambda)$ is approximately $(1-2\lambda)d+1$. Thus the first term in the above expression for $1-F(d,\lambda)$ behaves like $\beta^{\lambda d-1}((1-2\lambda)d+1)$ and the second term like $2^{-n(d,\lambda)}$. Therefore $\lim_{d\to\infty}(1-F(d,\lambda))^{1/d} = \max(\beta^\lambda, 2^{-(1-2\lambda)})$. If we solve the equation $\beta^\gamma = 2^{-(1-2\gamma)}$ for γ we obtain $\gamma = \frac{\ln 2}{2\ln 2 - \ln(\sqrt{3}/2)} \approx .452997$. For $\lambda < \gamma$, we have $\max(\beta^\lambda, 2^{-(1-2\lambda)}) = \beta^\lambda > \beta^\gamma$ and for $\lambda > \gamma$, we have $\max(\beta^\lambda, 2^{-(1-2\lambda)}) = 2^{-(1-2\lambda)} > 2^{-(1-2\gamma)} = \beta^\gamma$. This shows that $\limsup_{d\to\infty}(cm_d/2^{d-1})^{1/d} \leq \beta^\gamma$ and hence $\limsup_{d\to\infty} cm_d^{1/d} \leq 2\beta^\gamma \approx 1.87384$ as claimed. □

The following table illustrates our results. This table includes the curves of length 2, which are all Cohen-Macaulay and hence maximal, as well as the single curve $\{1\}$ of length 1, which corresponds to the projective line. (In talks we have sometimes omitted the curves of length 1 and 2, resulting in slightly different values.) In the table e_d is the number of maximal curves of degree d.

d	c_d	cm_d	ncm_d	e_d	r_d
1	1	1	0	1	
2	1	1	0	1	
3	3	3	0	3	1
4	6	5	1	5	.898
5	15	12	3	12	.918
6	27	16	11	16	.841
7	63	37	26	38	.872
8	120	51	69	52	.821
9	252	97	155	106	.827
10	495	142	353	146	.799
11	1023	257	766	294	.801
12	2010	359	1651	388	.774
13	4095	647	3448	765	.778
14	8127	920	7207	1011	.757
15	16365	1605	14760	1911	.761
16	32640	2266	30374	2556	.743
17	65535	3795	61740	4769	.743
18	130788	5410	125378	6169	.730

Since the growth of both c_d and cm_d is exponential it is natural to calculate the ratio $r_d = \log(cm_d)/\log(c_d)$. This is plotted below, with horizontal axis representing d.



If we represent cm_d in the form $2^{\alpha d}$ and approximate c_d by 2^{d-1} then $r_d = \alpha d/(d-1)$ (or asymptotically α). Thus if d is large we can think of the vertical scale as being α. The trend is clearly downwards, and it appears that α will approach a limit as $d \to \infty$. There is an oscillation between odd and even d that we have not yet explained. The data suggests that the upper bound .906 for the limit of α obtained in Theorem 4.4 is far too large. The value .5 obtained in Corollary 4.3 is a lower bound for the limit, and could even be the exact value.

5. Maximal curves of large length are Cohen-Macaulay.

In this section we prove that maximal curves of degree d with length greater than $d/2$ are Cohen-Macaulay. The case where d is even and the length is equal to $d/2$ is more delicate, and is reserved for the next section. First we prove a preliminary lemma.

Lemma 5.1. *Let $a \geq 1$ be an integer and let X be a subset of $[1, a-1]$. If $|X| \geq a/2$ then $a = x_1 + x_2$ with $x_1, x_2 \in X$ (we allow $x_1 = x_2$).*

Proof. If a is odd then we have $\lfloor a/2 \rfloor$ equations

$$\begin{aligned} a &= 1 + (a-1) \\ &= 2 + (a-2) \\ &= 3 + (a-3) \\ &\vdots \\ &= \lfloor a/2 \rfloor + \lceil a/2 \rceil \end{aligned}$$

Since $|X| \geq a/2$ and a is odd we have $|X| > \lfloor a/2 \rfloor$ so X must contain at least one of the pairs $\{1, a-1\}, \{2, a-2\} \ldots \{\lfloor a/2 \rfloor, \lceil a/2 \rceil\}$ from which the Lemma follows.

If a is even then we have $a/2$ equations

$$\begin{aligned} a &= 1 + (a-1) \\ &= 2 + (a-2) \\ &= 3 + (a-3) \\ &\vdots \\ &= ((a/2)-1) + ((a/2)+1) \\ &= 2(a/2) \end{aligned}$$

If $a/2 \in X$ then the Lemma holds. If $a/2 \notin X$ then X contains one of the pairs $\{1, a-1\}, \{2, a-2\} \ldots \{(a/2)-1, (a/2)+1\}$ and again the Lemma follows. $\square$

Theorem 5.2. *Let d be a positive integer greater than 1. If d is even write $d = 2n$ and if d is odd write $d = 2n+1$ (so that $n = \lfloor d/2 \rfloor$). Then a maximal curve $\mathscr{S}$ of degree d and length greater than n is Cohen-Macaulay.*

Proof. The proof is by Corollary 2.13. Maximality holds by assumption. We now prove that if $a \in [1, d-1]$ then either $a \in \Gamma$ or $d-a \in \hat{\Gamma}$, which is 2.13(a). So let $a \in [1, d-1]$. If $a \in \mathscr{S}$ (equivalently $d-a \in \hat{\mathscr{S}}$) then $a \in \Gamma$. If $a \notin \mathscr{S}$ then $\mathscr{S}$ contains at least n elements (omitting d itself) which are divided into n_1 in the interval $[1, a-1]$ and n_2 in the interval $[a+1, d-1]$, $n \le n_1 + n_2$. (Either of these intervals might be empty, in which case the corresponding n_i will be 0). Suppose that $n_1 < a/2$ and $n_2 < (d-a)/2$. Then adding these inequalites we get $n \le n_1 + n_2 < (d/2)$. If d is even then $n = d/2$ and we have the contradiction $n < n$. If d is odd we have $d/2 = n + 1/2$ which does not yet yield a contradiction, so we have to look more closely at the inequalities. First consider the case d odd and a even. Then $a/2$ is an integer so the inequality $n_1 < a/2$ is equivalent to $n_1 \le (a/2) - 1$. Adding this to the inequality $n_2 < (d-a)/2$ we get $n < (d/2) - 1 = n - (1/2)$ which is a contradiction. Now consider the case d odd and a odd. Then $d-a$ is even so $(d-a)/2$ is an integer and the inequality $n_2 < (d-a)/2$ is equivalent to $n_2 \le ((d-a)/2) - 1$. Adding the two inequalities we obtain $n \le n_1 + n_2 < (d/2) - 1 = n - (1/2)$ which again is a contradiction. In all cases we must have either $n_1 \ge a/2$ or $n_2 \ge (d-a)/2$. In the case $n_1 \ge a/2$ we have $a = x_1 + x_2$ with $x_i \in \mathscr{S}$ so $a \in \Gamma$ (with $\operatorname{ord}_{\mathscr{S}}(a) = 2$) by Lemma 5.1. In the case $n_2 \ge (d-a)/2$, $\hat{\mathscr{S}}$ contains at least $(d-a)/2$ elements in the interval $[1, d-a-1]$ and by Lemma 5.1 we have that $d-a \in \hat{\Gamma}$ with $\operatorname{ord}_{\hat{\mathscr{S}}}(d-a) = 2$. This completes the proof of assumption 2.13(a).

Note that we did not use $\mathscr{S}$ maximal in order to prove that 2.13(a) holds. We have shown so far that if $a \in [1, d-1]$ then either $a \in \Gamma$ with $\operatorname{ord}_{\mathscr{S}}(a) \leq 2$ or $d-a \in \hat{\Gamma}$ with $\operatorname{ord}_{\hat{\mathscr{S}}}(d-a) \leq 2$. In order to show hat 2.13(b) holds, it suffices to observe that by maximality if $\operatorname{ord}_{\mathscr{S}}(a) = 2$ then $d-a \notin \hat{\Gamma}$ and that if $\operatorname{ord}_{\hat{\mathscr{S}}}(d-a) = 2$ then $a \notin \Gamma$ (so as to rule out, for example, $\operatorname{ord}_{\mathscr{S}}(a) = 2$ and $\operatorname{ord}_{\hat{\mathscr{S}}}(d-a) > 2$). □

Example 5.3. If d is odd and the length of $\mathscr{S}$ is n then $\mathscr{S}$ need not be Cohen-Macaulay. For example if $d = 7, n = 3$ then $\mathscr{S} = \{2, 5, 7\}$ is maximal but not Cohen-Macaulay.

6. The critical case $d = 2n$ and $|\mathscr{S}| = n$.

The notation $d = 2n, n \geq 2$ and $\mathscr{S}$ a curve with $|\mathscr{S}| = n$ will apply throughout this section, except for the final remark. (The case $n = 1$ is vacuous.) We show that if $\mathscr{S}$ maximal then $\mathscr{S}$ is still Cohen-Macaulay, even though Corollary 2.13 need not apply.

Definition 6.1. Let $\mathscr{S}$ be a (not necessarily maximal) curve of degree d with associated numerical semigroups Γ and $\hat{\Gamma}$. If $a \in [1, d]$ we say that a is missing if $a \notin \Gamma$ and $d - a \notin \hat{\Gamma}$.

Lemma 6.2. *Let $\mathscr{S}$ be a maximal curve of degree d with $d = 2n$ and $|\mathscr{S}| = n$. Let $a \in [1, d]$, a even. Then a cannot be missing. Furthermore if $a \notin \mathscr{S}$ but $a \in \Gamma$ then $\operatorname{ord}_{\mathscr{S}}(a) = 2$ and $d - a \notin \hat{\Gamma}$. Dually if $a \in \hat{\Gamma}$ then $\operatorname{ord}_{\hat{\mathscr{S}}}(a) \leq 2$.*

Proof. Clearly d cannot be missing. If $a \in [1, d-1]$ is even and $a/2 \in \mathscr{S}$ then $a \in \Gamma$ so a is not missing. Similarly if $(d-a)/2 \in \hat{\mathscr{S}}$ (equivalently $d - ((d-a)/2) = (d+a)/2 \in \mathscr{S}$) then $d - a \in \hat{\Gamma}$ and again a is not missing. Also if $a \in \mathscr{S}$ or equivalently $d - a \in \hat{\mathscr{S}}$ then a is not missing and $\operatorname{ord}_{\mathscr{S}}(a) = 1$. Otherwise $a \notin \mathscr{S}$ (equivalently $d - a \notin \hat{\mathscr{S}}$) so $\mathscr{S}$ contains $n-1$ elements in $[1, d-1] \setminus \{a/2, a, (d+a)/2\}$ (which contains $d-4$ elements). The set $[1, d-1] \setminus \{a/2, a, (d+a)/2\}$ is partitioned into $(d-4)/2 = n-2$ pairs of elements $\{(1, a-1), (2, a-2), \ldots, ((a/2)-1, (a/2)+1), (a+1, d-1), (a+2, d-2), \ldots, (((d+a)/2)-1, ((d+a)/2)+1)\}$. Both elements of at least one of these pairs thus must be in $\mathscr{S}$. If $\{i, a-i\} \subset \mathscr{S}$ (for some i, $1 \leq i \leq (a/2)-1$) then $a \in \Gamma$ and $\operatorname{ord}_{\mathscr{S}}(a) = 2$. (In this case we cannot have $d - a \in \hat{\Gamma}$ because then we would have both $a \in \Gamma$ and $d - a \in \hat{\Gamma}$ which by maximality of $\mathscr{S}$ and Lemma 2.4 would imply $a \in \mathscr{S}$, a contradiction.) If $\{a+i, d-i\} \subset \mathscr{S}$ (for some i, $1 \leq i \leq (d/2) - (a/2) - 1$) then $\{d-a-i, i\} \subset \hat{\mathscr{S}}$ so $d - a \in \hat{\Gamma}$ and again a is not missing. □

Lemma 6.3. *If $d \geq 8$ (d even) and $a \in [1, d]$ is odd, then there exists $\mathscr{S}$ of degree d with $|\mathscr{S}| = n$ for which a is missing.*

Proof. Take $\mathscr{S} = \{\lceil a/2 \rceil, \dots, a-1, a+1, \dots, \lfloor (d+a)/2 \rfloor\} \cup \{d\}$. Then $a \notin \Gamma$ because the sum of any two of $\{\lceil a/2 \rceil, \dots, a-1\}$ is greater than a. Similarly $d - a \notin \hat{\Gamma}$, so a is missing. $\square$

Note that if $d = 6$ and $a = 3$ then the construction above yields $\mathscr{S} = \{2, 4, 6\}$ which is not a curve because the gcd is greater than one. If $d = 6$ then 3 cannot be missing.

Lemma 6.4. *Given $\mathscr{T} \subseteq [1, 2k]$, $|\mathscr{T}| = k$, $\mathscr{T} \neq \{2, 4, 6, \dots, 2k\}$, and $2k + 1 \notin \Gamma_{\mathscr{T}}$ (the semigroup generated by $\mathscr{T}$), then $\{x \in \mathbb{N} | x > 2k + 1\} \subseteq \Gamma_{\mathscr{T}}$.*

Proof. Note that since $|\mathscr{T}| = k$ and $2k + 1 \notin \Gamma_{\mathscr{T}} = \Gamma$, there is exactly one element of $\mathscr{T}$ in each pair $\{i, 2k + 1 - i\}$, $1 \leq i \leq k$. Let $m = \min(\mathscr{T})$. By the previous remark, we must have $\{2k + 1 - i | 1 \leq i < m\} \subseteq \mathscr{T}$. Since $m \in \mathscr{T}$, we have $\{2k + m + 1 - i | 1 \leq i < m\} \subseteq \Gamma$. If we can show that $2k + m + 1 \in \Gamma$, we will be done since we will have m consecutive values $\{2k+2, \dots, 2k+m+1\}$ in Γ and repeatedly adding m will give us the desired result.

If $m = 1$, then $\Gamma = \mathbb{N}$ and the result follows vacuously.

If $m = 2$, since $\mathscr{T} \neq \{2, 4, 6, \dots, 2k\}$ there must be an odd integer $2\ell + 1 \in \mathscr{T}$, $\ell < k$. In this case, $\{i \in \mathbb{N} | i \geq 2\ell\} \subseteq \Gamma$ and again the result follows vacuously.

Suppose that $m > 2$. Let α be the largest integer such that $\alpha m \in \mathscr{T}$ and for all $x < \alpha m$, $x \in \mathscr{T}$ implies that m divides x. Since $2k/m < k$, $\mathscr{T}$ must contain some element that is not a multiple of m. By the definition of α there must therefore be some β with $0 < \beta < m$ such that $\alpha m + \beta \in \mathscr{T}$ but $(\alpha - 1)m + \beta \notin \mathscr{T}$. Since $(\alpha - 1)m + \beta \notin \mathscr{T}$ we must have $2k + 1 - (\alpha - 1)m - \beta \in \mathscr{T}$. But then $(2k + 1 - (\alpha - 1)m - \beta) + (\alpha m + \beta) = 2k + m + 1 \in \Gamma$ and we are done. $\square$

Lemma 6.5. *If $\mathscr{S}$ is a (not necessarily maximal) curve of degree $d = 2n$ and $|\mathscr{S}| = n$, then there cannot be two missing elements.*

Proof. Suppose $a, b \in [1, d - 1]$ are missing. That is, $a, b \notin \Gamma$ and $d - a, d - b \notin \hat{\Gamma}$. Let $a = 2s + 1$, $b = 2t + 1$, and without loss of generality, assume that $s < t$. Let $\mathscr{T}_1 = \mathscr{S} \cap [1, 2s]$ and $\mathscr{T}_1' = \mathscr{S} \cap [2s + 2, 2n - 1]$. Note that since $a \notin \Gamma$, $|\mathscr{T}_1| \leq s$ and since

$d - a \notin \hat{\Gamma}$, $|\mathscr{T}_1'| \leq n - s - 1$. But as $\mathscr{T}_1 \cup \mathscr{T}_1' = \mathscr{S}\backslash\{2n\}$ and $|\mathscr{S}\backslash\{2n\}| = n - 1$, we must have $|\mathscr{T}_1| = s$ and $|\mathscr{T}_1'| = n - s - 1$. If we apply the previous lemma with $\mathscr{T} = \mathscr{T}_1$ and $k = s$, we see that unless $\mathscr{T}_1 = \{2, 4, 6, \ldots, 2s\}$, $b \in \Gamma_{\mathscr{T}} \subseteq \Gamma_{\mathscr{S}}$ and we would have a contradiction. Therefore $\mathscr{S} \cap [1, 2s] = \{2, 4, 6, \ldots, 2s\}$. A similar argument using $\mathscr{T} = 2n - (\mathscr{S} \cap [2t+2, 2n-1])$ and $k = n-t-1$ shows that we must have $\mathscr{T}' = \mathscr{S} \cap [2t+2, 2n-1] = \{2t+2, 2t+4, 2t+6, \ldots, 2n-2\}$.

Now suppose some element of the set $\mathscr{S} \cap [2s + 2, 2t]\}$ were odd, say $2\ell + 1 \in \mathscr{S}$ with $s < \ell < t$. Since $2 \in \mathscr{S}$, we would have $2(t - \ell) + (2\ell + 1) = 2t + 1 \in \Gamma_{\mathscr{S}}$ contradicting the fact that $b \notin \Gamma_{\mathscr{S}}$.

We have shown that all of the elements of $\mathscr{S}$ are even, but this contradicts the underlying assumption that the group generated by $\mathscr{S}$ is $\mathbb{Z}$. □

Lemma 6.6. *Suppose that $\mathscr{S}$ is a curve of degree $d = 2n$ with $|\mathscr{S}| = n$ and one element missing. Then $\mathscr{S}$ is maximal.*

Proof. Suppose that $a \in [1, d-1]$ is missing. Necessarily a is odd. As in the proof of Lemma 6.5 half the elements in $[1, a-1]$ belong to $\mathscr{S}$ and half the elements of $[a+1, d-1]$ belong to $\mathscr{S}$. But since $a \notin \Gamma$ at most half the elements of $[1, a-1]$ belong to Γ. Therefore

(a) $\qquad \mathscr{S} \cap [1, a-1] = \Gamma \cap [1, a-1]$.

Similarly

(b) $\qquad (d - \hat{\mathscr{S}}) \cap [a+1, d-1] = (d - \hat{\Gamma}) \cap [a+1, d-1]$.

Now suppose that $b \in \Gamma$ and $d - b \in \hat{\Gamma}$. If $b < a$ then, by (a), $b \in \mathscr{S}$. If $b > a$ then by (b) $d - b \in \hat{\mathscr{S}}$ and again $b \in \mathscr{S}$. So $\mathscr{S}$ is maximal by Lemma 2.4. □

Lemma 6.7. *Suppose that $\mathscr{S}$ is a curve of degree $d = 2n$ with $|\mathscr{S}| = n$ and one element $a \in [1, d-1]$ missing. Then $[a+1, d] \subseteq \Gamma$.*

Proof. If $b \in [a+1, d-1], b \notin \Gamma$ then by assumption b is not missing, so $d - b \in \hat{\Gamma}$. But then by (b) in the proof of Lemma 6.6, $d - b \in \hat{\mathscr{S}}$, so $b \in \mathscr{S}$, contradiction. □

Lemma 6.8. *Suppose that $\mathscr{S}$ is a maximal curve of degree $d = 2n$ with $|\mathscr{S}| = n$. If $1 \notin \mathscr{S}$ then* $\mathrm{ord}_{\mathscr{S}}(b) \leq 2$ *for all $b \in \Gamma \cap [1, d]$.*

Proof. If b is even, the claim holds by Lemma 6.2. So we need only consider b odd, $b \in \Gamma\backslash\mathscr{S}$. By maximality this will imply that $d - b \notin \hat{\Gamma}$.

If $1 \in \hat{\mathscr{S}}$ then $\mathscr{S} = \Gamma \cap [1, d]$ by Lemma 2.5, so $b \in \mathscr{S}$, contradiction. Hence we need only consider the case $1 \notin \hat{\mathscr{S}}$ (equivalently $d - 1 \notin \mathscr{S}$).

We have $n-1$ pairs $\{i, b-i\}, 1 \le i \le \lfloor b/2 \rfloor$ and $\{b+i, d-i\}, 1 \le i \le \lfloor (d-b)/2 \rfloor$ which contain $n-1$ elements of $\mathscr{S}$. If one of these pairs has neither its elements in $\mathscr{S}$ then one of the pairs must have both of its elements in $\mathscr{S}$. This pair cannot be of the form $\{b+i, d-i\}$ because that would imply $d-b \in \hat{\Gamma}$. Therefore the pair must be of the form $\{i, b-i\}$ which implies that $\mathrm{ord}_{\mathscr{S}}(b) = 2$. If $d-2 \in \mathscr{S}$, equivalently $2 \in \hat{\mathscr{S}}$ then $\hat{\Gamma}$ contains all the even integers in $[1, d]$. Because $\mathscr{S}$ is maximal we have, by Lemma 2.4, that all even integers in $\Gamma \cap [1, d]$ lie in $\mathscr{S}$. Since $b \in \Gamma$ we can write b as the sum of an odd element of $\mathscr{S}$ and an even element of $\Gamma \cap [1, d]$. The latter is in $\mathscr{S}$ so $\mathrm{ord}_{\mathscr{S}}(b) = 2$. Hence we need consider only the case where $d-2 \notin \mathscr{S}$.

Suppose first that $b < d-1$ so that there are pairs of the type $\{b+i, d-i\}$. If no pair of the form $\{b+i, d-i\}$ has both elements not in $\mathscr{S}$ then (because we assume that $2 \notin \hat{\Gamma}$) we can apply Lemma 6.4 with $k = (d-b-1)/2$ to $\hat{\mathscr{S}} \cap [1, d-b-1]$ yielding $[d-b+1, d] \subset \hat{\Gamma}$. By maximality, this implies that $\mathscr{S} \cap [1, b-1] = \Gamma \cap [1, b-1]$ from which it follows that if $b \in \Gamma \backslash \mathscr{S}$ then $\mathrm{ord}_{\mathscr{S}}(b) = 2$. Now consider $b = d-1$. Since we assume that $d-2 \notin \mathscr{S}$, $\{1, d-2\}$ is our missing pair, completing the proof. □

Theorem 6.9. *Suppose that $\mathscr{S}$ is a curve of degree $d = 2n$ with $|\mathscr{S}| = n$ and one element $a \in [1, d-1]$ missing. Then $\mathscr{S}$ is Cohen-Macaulay.*

Proof. First note that $\mathscr{S}$ is maximal by Lemma 6.6 (this is needed for Lemma 6.8, which is used several times in the proof). Suppose that $a \in [1, d-1]$ is missing. The element a is necessarily odd. Also $1 \notin \mathscr{S}$ and $1 \notin \hat{\mathscr{S}}$ so Lemma 6.8 applies to both $\mathscr{S}$ and $\hat{\mathscr{S}}$. Thus $\mathrm{ord}_{\mathscr{S}}(b) \le 2$ for all $b \in \Gamma \cap [1, d]$ and $\mathrm{ord}_{\hat{\mathscr{S}}}(b) \le 2$ for all $b \in \hat{\Gamma} \cap [1, d]$. Suppose that $(A, B) \in S'$. If we show that $(A, B) \in S$ then $\mathscr{S}$ is Cohen-Macaulay. Write $(A, B) = (\alpha, \beta) + (c_1 d, c_2 d)$ with $0 \le \alpha < d,\ 0 \le \beta < d,\ c_1 \ge 0,\ c_2 \ge 0$. If $(\alpha, \beta) \ne (d-a, a)$ then $(A, B) \in S$ by Corollary 2.12. So suppose that $(\alpha, \beta) = (d-a, a)$. Since a is missing, $\alpha \notin \hat{\Gamma}$ and $\beta \notin \Gamma$. Therefore we must have $c_1 \ge 1$ and $c_2 \ge 1$, and it suffices to prove that $(2d-a, d+a) \in S$, equivalently $\mathrm{ord}_{\mathscr{S}}(d+a) \le 3$ (actually $\mathrm{ord}_{\mathscr{S}}(d+a) \le 2$ is ruled out because $d-a \notin \hat{\Gamma}$ so we must really have $\mathrm{ord}_{\mathscr{S}}(d+a) = 3$). First consider the case $a < d-1$. As in the proof of Lemma 6.5 there is some element γ of $\mathscr{S}$ in the interval $[a+1, d-1]$. Then $a+d-\gamma \in [a+1, d-1]$ so $a+d-\gamma \in \Gamma$ by Lemma 6.7. By Lemma 6.8, $\mathrm{ord}_{\mathscr{S}}(a+d-\gamma) \le 2$, hence $\mathrm{ord}_{\mathscr{S}}(a+d) = \mathrm{ord}_{\mathscr{S}}(\gamma + (a+d-\gamma) \le \mathrm{ord}_{\mathscr{S}}(\gamma) + \mathrm{ord}_{\mathscr{S}}(a+d-\gamma) \le 1+2 = 3$. Now suppose that $a = d-1$. Half the elements of $[1, d-2]$ are in $\mathscr{S}$, no two of which add up to a, so there must be one element of $\mathscr{S}$ in each pair $\{i, a-i\}$, $1 \leq i < a/2$. But $1 \notin S$ so we must have $d-2 \in \mathscr{S}$. By Lemma 6.4 applied to $\mathscr{S} \cap [1, d-2]$ with $2k = d-2$ we have $d+1 \in \Gamma$. By Lemma 6.6 assertion (a) we have $\mathscr{S} = \Gamma \cap [1, d]$ from which it follows that $\mathrm{ord}_{\mathscr{S}}(d+1) = 2$. Since $2d-1 = (d-2)+(d+1)$ we have $\mathrm{ord}_{\mathscr{S}}(2d-1) \leq 3$ as desired. □

There remains only the case where $1 \in \mathscr{S}$ (the case $1 \in \hat{\mathscr{S}}$ being dual).

Lemma 6.10. *Suppose that $\mathscr{S}$ is a maximal curve of degree $d = 2n$ with $|\mathscr{S}| = n$ and $1 \in \mathscr{S}$. Then $\mathrm{ord}_{\mathscr{S}}(a) \leq 2$ for $a \in [2, d-2]$, and $\mathrm{ord}_{\mathscr{S}}(d-1) \leq 3$.*

Proof. By Lemma 6.2, $\mathrm{ord}_{\mathscr{S}}(a) \leq 2$ if a is even. So suppose that a is odd, $a < d-2$. If $\mathrm{ord}_{\mathscr{S}}(a) > 2$, by Lemma 5.1 at most half the elements of $[1, a-1]$ belong to $\mathscr{S}$. Since $a \notin \mathscr{S}$, $d-a \notin \hat{\mathscr{S}}$. Then, by Lemma 2.5, $d-a \notin \hat{\Gamma}$. From this it follows that at most half the elements of $[a+1, d-1]$ belong to $d-\hat{\mathscr{S}}$, and hence to $\mathscr{S}$. Since $|\mathscr{S}| = d/2$ we now have that half the elements of $[1, a-1]$ and half the elements of $[a+1, d-1]$ belong to $\mathscr{S}$. By Lemma 2.5, $\hat{\mathscr{S}} = \hat{\Gamma} \cap]1, d]$. But by Lemma 6.4 applied with $k = (d-a-1)/2$ to $\hat{\mathscr{S}} \cap [1, d-a-1]$, we conclude that (since $\hat{\mathscr{S}} \neq \{2, 4, 6, \ldots, d-2, d\}$) $\hat{\mathscr{S}}$ contains all elements of $[d-a+1, d-1]$ (note that this interval contains at least two elements), which will yield more than n elements in $\hat{\mathscr{S}}$, contradiction. Thus $\mathrm{ord}_{\mathscr{S}}(a) \leq 2$ for all a, $2 \leq a \leq d-2$. Finally $\mathrm{ord}_{\mathscr{S}}(d-2) \leq 2$ and $1 \in \mathscr{S}$ so $\mathrm{ord}_{\mathscr{S}}(d-1) \leq 3$. □

Lemma 6.11. *Suppose that $\mathscr{S}$ is a maximal curve of degree $d = 2n$ with $|\mathscr{S}| = n$, $1 \in \mathscr{S}$ and $\mathrm{ord}_{\mathscr{S}}(d-1) = 3$. Then $d+1 \notin \hat{\Gamma}$.*

Proof. Since $\mathrm{ord}_{\mathscr{S}}(d-1) > 2$ we have that $(d+1, d-1) \notin S$. Therefore if $d+1 \in \hat{\Gamma}$ then $\mathrm{ord}_{\hat{\mathscr{S}}}(d+1) \geq 3$. But $1 \notin \hat{\mathscr{S}}$ and $\hat{\mathscr{S}} \cap [1, d] = \hat{\Gamma} \cap [1, d]$ from which it follows that if $d+1 \in \hat{\Gamma}$ then $\mathrm{ord}_{\hat{\mathscr{S}}}(d+1) = 2$, contradiction. □

Lemma 6.12. *Suppose that $\mathscr{S}$ is a maximal curve of degree $d = 2n$ with $|\mathscr{S}| = n$ and $1 \in \mathscr{S}$. Then $\mathscr{S}$ is Cohen-Macaulay.*

Proof. First note that $\hat{\mathscr{S}} = \hat{\Gamma} \cap [1, d]$ by Lemma 2.5, i.e. $\mathrm{ord}_{\hat{\mathscr{S}}}(x) = 1$ for all $x \in [1, d] \cap \hat{\Gamma}$, so in applying Corollary 2.13 we need consider only $\mathrm{ord}_{\mathscr{S}}(x)$. If $\mathrm{ord}_{\mathscr{S}}(d-1) < 3$ it follows from Lemma 6.10 and Corollary 2.13 that $\mathscr{S}$ is Cohen-Macaulay. So suppose that $\mathrm{ord}_{\mathscr{S}}(d-1) = 3$. If $\mathscr{S}$ is not Cohen-Macaulay, then there exists $(a, b) \in S' \backslash S$. As in the proof of Corollary 2.13 write $(a, b) = (\alpha, \beta) + (c_1 d, c_2 d)$ with $0 \leq \alpha <$

$d,\ 0 \leq \beta < d,\ c_1 \geq 0, c_2 \geq 0$. If $(\alpha, \beta) \neq (1, d-1)$ then $(a, b) \in \mathscr{S}$ by Corollary 2.12. So suppose that $(\alpha, \beta) = (1, d-1)$. Since, by Lemma 6.11, $d+1 \notin \hat{\Gamma}$ we have that $c_1 \geq 2$. But $c_1 = 2$ corresponds to $(2d+1, d-1)$, which is in S because $\mathrm{ord}_{\mathscr{S}}(d-1) = 3$. Therefore also $(a, b) \in S$, contradiction. □

Theorem 6.13. *Suppose that $\mathscr{S}$ is a maximal curve of degree $d = 2n$ with $|\mathscr{S}| = n$. Then $\mathscr{S}$ is Cohen-Macaulay.*

Proof. If $1 \in \mathscr{S}$ or $1 \in \hat{\mathscr{S}}$ this follows from Lemma 6.12. If $1 \notin \mathscr{S}$, $1 \notin \hat{\mathscr{S}}$ and one element is missing this follows from Theorem 6.9. If $1 \notin \mathscr{S}$, $1 \notin \hat{\mathscr{S}}$ and no elements are missing this follows from Lemma 6.8 and Theorem 2.13. □

Remark 6.14. If $\mathscr{S}$ is a curve of degree d and $|\mathscr{S}| = n$ we have done some calculation (without proofs) when n is just a bit less than $d/2$ (i.e. $d = 2n + i$ for $i > 0$ small compared to n) that suggests the following (extending some of our theorems when $i = 0$). At most $i + 1$ elements are missing, and for fixed n and $d = 2n + i$ the number of curves decreases rapidly as the number of missing elements increases (for fixed i the more so as n increases). If fewer than i elements are missing most curves are not maximal, but most of those that are maximal are Cohen-Macaulay. If at least i elements are missing then most (perhaps all) curves are maximal, and most of the maximal curves are Cohen-Macaulay. For example if $n = 8$ and $d = 18$ (so that $i = 2$) then the number of curves with 0,1,2,3,4 elements missing are respectively 18754, 622, 60, 4, 0, the number that are maximal are respectively 679, 340, 60, 4, 0 and the number that are Cohen-Macaulay are respectively 671, 340, 60, 4, 0. However these are only initial observations and more extensive calculation (or proofs!) will likely modify them.

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