Ideal Class Groups of Subintegral Extensions

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Let $A \subseteq B$ be commutative rings. Then $A \subseteq B$ is subintegral $\Leftrightarrow$

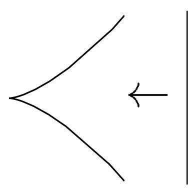
(i) B is integral over A.

(ii) $\mathrm{Spec}B \to \mathrm{Spec}A$ ($Q \to Q \cap A$) is a bijection and the induced homomorphism

$$\kappa(\frac{A}{Q \cap A}) \to \kappa(\frac{B}{Q})$$

is an isomorphism for all $Q \in \mathrm{Spec}B$.

Example: $k[t^2, t^3] \subseteq k[t]$ is subintegral.



More generally $A \subseteq A[b]$, with $b^2, b^3 \in A$ is called *elementary subintegral.* An elementary subintegral extension is subintegral.

Exercise: Give an example of an extension $A \subseteq B$ that satisfies (ii) but not (i).

Basic Properties of Subintegral Extensions:

1. Given $A \subseteq B$ there is a largest subextension $A \subseteq {}^+_B A \subseteq B$ such that $A \subseteq {}^+_B A$ is subintegral.

2. $b \in {}^+_B A \Leftrightarrow \exists$ a chain $A \subseteq A_1 \subseteq A_2 \ldots \subseteq A_n \subseteq B$ such that $b \in A_n$ and $A_i \subseteq A_{i+1}$ is elementary subintegral.

3. Given $A \subseteq B \subseteq C$ then $A \subseteq C$ is subintegral $\Leftrightarrow$ both $A \subseteq B$ and $B \subseteq C$ are subintegral.

4. If S is a multiplicative subset of A and $A \subseteq B$ is subintegral so is $A_S \subseteq B_S$.

Definitions:

1. ${}^+_B A$ is called the seminormalization of A in B.

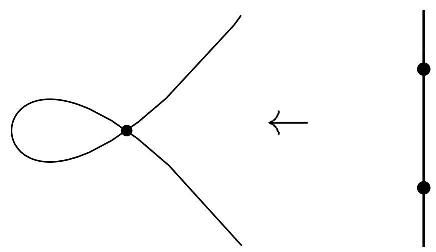
2. A is defined to be seminormal in B if ${}^+_B A = A$. Then we have ${}^+_B A$ is seminormal in B.

3. (absolute notions) Suppose (for simplicity) that A is reduced Noetherian (so that the total ring of fractions K of A is a finite product of fields). Then ${}^+A := {}^+_K A$ is the seminormalizaton of A and A is defined to be seminormal if $A = {}^+A$.

Example. $A = \{f \in k[t] \mid f(1) = f(-1)\} \subseteq B = k[t]$ (k a field of characteristic $\neq 2$).Then ${}^+_B A = ({}^+A =)\ A$ i.e. A is seminormal in B (and seminormal in the absolute sense).

Proof. Suppose $f \in B, f^2, f^3 \in A$. Then $f(-1)^3 = f(1)^3, f(-1)^2 = f(1)^2$. Dividing we get $f(-1) = f(1)$ so $f \in A$. A has no elementary subintegral extensions in B so ${}^+_B A = A$.$\square$

Note that Spec$A$ is the line with two points identified so A is the affine co-ordinate ring of a node.



Reference Swan "On seminormality" J.Alg.67(1980) 210-229.

Why study seminormality/subintegrality?

1. Motivation from K-Theory:

$$\mathrm{Pic}A \to \mathrm{Pic}A[T]$$

is an isomorphism $\Leftrightarrow A_{\mathrm{red}}$ is seminormal.

2. $\mathrm{Spec}^+A \to \mathrm{Spec}A$ can be thought of as the maximal resolution of singularities without separating points.

3. Generic projections are seminormal: let X be an algebraic variety (in some projective space). $\dim X = r$, X non-singular. We can project X into $\mathbb{P}^{2r+1}$ isomorphically. For a suitable range of values $m < 2r+1$ (with suitable hypotheses) we can project X into $\mathbb{P}^m$ so that the image of X is seminormal. For example if X is a curve, $r = 1, 2r+1 = 3$. We can project X isomorphically into $\mathbb{P}^3$. We can project X into $\mathbb{P}^2$ with only nodes as singularities (which is equivalent to the image of X being seminormal).

Definition. Let $A \subseteq B$ be any extension of commutative rings. An A-submodule M of B is defined to be invertible if there exists an A-submodule N of B such that $MN = A$. Then we define $\mathfrak{I}(A, B)$ to be the group of invertible A submodules of B.

Properties.

1. If $M \in \mathfrak{I}$ then M is a finitely generated projective A module of rank 1.

2. There is an exact sequence

$$0 \to A^* \to B^* \to \mathfrak{I}(A, B) \to \mathrm{Pic}(A) \to \mathrm{Pic}(B)$$

Theorem 1. (Roberts-Singh, Subintegrality, invertible modules and the Picard group, Comp.Math.85 (1993) 249-279.) *Let $A \subseteq B$ be subintegral, and suppose that $\mathbb{Q} \subset A$. Then there is an isomorphism*

$$B/A \xrightarrow{\xi} \mathfrak{I}(A,B)$$

defined by $b \to (e^{bT}A[[T]]) \cap B[T] \xrightarrow{T \to 1} \xi(b) \in \mathfrak{I}(A,B)$.

Example. If b is nilpotent then $\xi(b) = Ae^b$. If $b^n \in A$ for all $n \geq N$ then $\xi(b) = A(e_N(b), b^N)$, where $e_N(b) = \sum_{i=0}^{N-1} b^i/i!$.

In Theorem 1 we require that $(e^{bT}A[[T]]) \cap B[T] \in \mathfrak{I}(A[T], B[T])$. In order to prove this we showed that

Theorem 2. *Let $A \subseteq B$ be an extension of $\mathbb{Q}$-algebras and let $b \in B$. Then $A \subseteq A[b]$ is subintegral $\Leftrightarrow \exists c_1, \ldots, c_p \in B$ $(p \geq 0)$ such that $b^n + \sum_{i=1}^{p} \binom{n}{i} c_i b^{n-i} \in A$ for $n >> 0$.*

Example. If $p = 0$ then Theorem 2 says that if $b^n \in A$ for $n \geq N$ then $A \subseteq A[b]$ is subintegral. (The extension is easily written as a sequence of elementary subintegral extensions.)

Remark. Suppose that $f \in (e^{bT}A[[T]]) \cap B[T]$, so that $e^{-bT}f \in A[[T]]$. The expressions $b^n + \sum_{i=1}^{p} \binom{n}{i} c_i b^{n-i}$ in Theorem 2 arise as the coefficient of T^n in $e^{-bT}f \in A[[T]]$.

Another idea that arises in the proof of Theorem 1 is the following:

Lemma. (easy) If I is an ideal of A and B then the map of pairs $(A, B) \to (A/I, B/I)$ induces an isomorphism $\mathfrak{I}(A, B) \to \mathfrak{I}(A/I, B/I)$.

The lemma permits induction on Krull dimension.

Question: What happens if we do not have $\mathbb{Q} \subseteq A$?

The exponential series seems to play an essential role in Theorems 1 and 2. However we have been able to remove the dependence on $\mathbb{Q}$ by the suitable use of "divided powers". In $B[[T]]$ define a new multiplication by $T^n * T^m = \binom{m+n}{n} T^{m+n}$ and extend by linearity and completion to $B[[T]]$. Then $\mathrm{Exp}(b) := \sum_{n=0}^{\infty} b^n T^n$ satisfies $\mathrm{Exp}(b_1) * \mathrm{Exp}(b_2) = \mathrm{Exp}(b_1 + b_2)$ so Exp behaves like an exponential. Using this Theorem 2 turns out to characterize weak subintegrality rather than subintegrality.

Definition. Let $A \subseteq B$ be commutative rings. Then $A \subseteq B$ is weakly subintegral $\Leftrightarrow$

(i) B is integral over A.

(ii) $\mathrm{Spec}B \to \mathrm{Spec}A$ ($Q \to Q \cap A$) is a bijection and the induced homomorphism

$$\kappa(\frac{A}{Q\cap A}) \to \kappa(\frac{B}{Q})$$

is a purely inseparable field extension for all $Q \in \mathrm{Spec} B$.

For $\mathbb{Q}$-algebras subintegral and weakly subintegral are the same. However in general the two concepts differ:

Example. Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Then $\mathbb{F}_p[X^p] \subseteq \mathbb{F}_p[X]$ is weakly subintegral but not subintegral.

Then we have

Theorem. (Reid-Roberts-Singh, On weak subintegrality, JPAA 114(1996)93-109.) *Let $A \subseteq B$ be an extension of commutative rings and and let $b \in B$. Then $A \subseteq A[b]$ is weakly subintegral $\Leftrightarrow \exists c_1, \ldots, c_p \in B$ $(p \geq 0)$ such that $b^n + \sum_{i=1}^{p} \binom{n}{i} c_i b^{n-i} \in A$ for $n >> 0$.*

I do not know if there is a similar characterization of the subintegrality of $A \subseteq A[b]$ without assuming that $\mathbb{Q} \subseteq A$. The following example shows that ξ cannot be an isomorphism in general:

Example. Let $A = \mathbb{F}_2 \oplus t^3\mathbb{F}_2[t] \subseteq B = \mathbb{F}_2[t]$. Then $\mathfrak{I}(A,B) = \mathfrak{I}(A/t^3B, B/t^3B) = \mathfrak{I}(\mathbb{F}_2, \mathbb{F}_2[t]/t^3) =$

$(\mathbb{F}_2[t]/t^3)^*$ and in the latter $(1+t)^2 \neq 1$. Thus $\mathfrak{I}(A,B)$ is not killed by 2, hence cannot be isomorphic to B/A which is killed by 2.

Note that in the above example even though $\mathfrak{I}(A,B)$ is not a vector space over $\mathbb{F}_2$ it has two parameters, as does $B/A \cong \mathbb{F}_2^2$. (More precisely $\mathfrak{I}(A,B)$ has a composition series with quotients $\mathbb{F}_2$.) Thus $\mathfrak{I}(A,B)$ is the "same size" as B/A. This observation perhaps makes more plausible the following recent result of Singh:

Let $A = \oplus_{n\geq 0} A_n \subseteq B = \oplus_{n\geq 0} B_n$ be an extension of graded rings. Define a new ring A_* to be A as abelian group, with product defined by $f * g = \binom{m+n}{m} fg$ if $f \in A, \deg f = m, \deg g = n$. Similarly define B_*. Let $\mathcal{F}_d A := A + \oplus_{n\geq d} B_d$. This is a decreasing filtration of graded A-subalgebras of B with $\mathcal{F}_0 A = B$ and $\cap_{d\geq 0} \mathcal{F}_d A = A$. Clearly $\mathfrak{I}(A, \mathcal{F}_d A) \subseteq \mathfrak{I}(A,B)$ yielding a decreasing filtration of $\mathfrak{I}(A,B)$. Let

$$\operatorname{gr} \mathfrak{I}(A,B) = \bigoplus_{d\geq 0} \mathfrak{I}(A, \mathcal{F}_d A)/\mathfrak{I}(A, \mathcal{F}_{d+1} A)$$

be the associated graded group. Then we have

Theorem. (B.Singh, Subintegrality and invertible modules in graded rings, to appear) *Let $A \subseteq B$ be a subintegral extension of positively graded rings with $A_0 = B_0$. Then there is a commutative diagram of functorial group isomorphisms:*

$$\begin{array}{ccc} B/A & & \\ \zeta_{B/A} \downarrow & \searrow^{\eta_{B/A}} & \\ \mathfrak{I}(A_*, B_*) & \stackrel{\varphi_{B/A}}{\longleftarrow} & \operatorname{gr} \mathfrak{I}(A, B) \end{array}.$$

References. Some additional related papers are:

1. B.Dayton, The Picard group of a reduced G-algebra, J.P.A.A 59(1989)237-253

2. L.Reid, L.Roberts, B.Singh, Finiteness of subintegrality, In: Goerss-Jardine (eds) Algebraic K-Theory and Algebraic Topology, Kluwer(1993)223-227

3. L.Reid and L.Roberts, A new criterion for weak subintegrality, Comm.Alg.24(10)3335-3342(1996)

4. L.Roberts and B.Singh, Invertible modules and generic subintegrality, J.P.A.A. 95(1994)331-351.

5. B.Singh, The Picard group and subintegrality in positive characteristic. Comp.Math 85(1993)249-279.

6. H.Yanagahara, On an intrinsic definition of weakly normal rings, Kobe J.Math 2(1985)89-98.