Monomial Subrings and Systems of Subintegrality

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Definition 1. Let $A \subseteq B$ be commutative rings. Then $A \subseteq B$ is subintegral $\Leftrightarrow$

(i) B is integral over A.

(ii) The induced map Spec$B \to$ Spec$A$ is a bijection and the induced homomorphism

$$\kappa(\frac{A}{Q \cap A}) \to \kappa(\frac{B}{Q})$$

is an isomorphism for all $Q \in$ Spec$B$.

Definition 2. Let $A \subseteq B$ be commutative rings. Then $A \subseteq B$ is weakly subintegral $\Leftrightarrow$

(i) B is integral over A.

(ii) The induced map Spec$B \to$ Spec$A$ is a bijection and the induced homomorphism

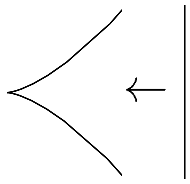
$$\kappa(\frac{A}{Q \cap A}) \to \kappa(\frac{B}{Q})$$

is a purely inseparable field extension for all $Q \in$ Spec$B$.

Obviously subintegral $\Rightarrow$ weakly subintegral, and if $\mathbb{Q} \subset A$ then subintegral $\Leftrightarrow$ weakly subintegral.

Examples 3.

1. $k[t^2, t^3] \subseteq k[t]$ is subintegral. (k any commutative ring.)



2. $\mathbb{F}_2[t^2] \subset \mathbb{F}_2[t]$ is weakly subintegral but not subintegral.

Facts 4. Given an arbitrary extension $A \subset B$ of commutative rings

1. There is a largest subextension ${}^+_BA$, $A \subseteq {}^+_BA \subseteq B$ such that $A \subseteq {}^+_BA$ is subintegral. ${}^+_BA$ is called the seminormalization of A in B, and A is seminormal in B if ${}^+_BA = A$.

2. There is a largest subextension ${}^*_BA$, $A \subseteq {}^*_BA \subseteq B$ such that $A \subseteq {}^*_BA$ is weakly subintegral. ${}^*_BA$ is called the weak normalization of A in B, and A is weakly normal in B if ${}^*_BA = A$.

3. An elementary subintegral extension of commutative rings is an extension of the form $R \subseteq R[x]$ where $x^2, x^3 \in R$. Then we have that $x \in {}^+_B A \Leftrightarrow$ there is a sequence of extensions $A = A_0 \subset A_1 \subset \cdots \subset A_d$ with $x \in A_d$ and $A_{i-1} \subset A_i$ elementary subintegral $(1 \leq i \leq d)$.

4. An elementary weakly subintegral extension is an extension of the form $R \subseteq R[x]$ where $x^p, px \in R$ for some prime p. Then we have that $x \in {}^*_B A \Leftrightarrow$ there is a sequence of extensions $A = A_0 \subset A_1 \subset \cdots \subset A_d$ with $x \in A_d$ and $A_{i-1} \subset A_i$ either elementary subintegral or elementary weakly subintegral $(1 \leq i \leq d)$.

The extension $\mathbb{Z}[t^2, 2t] \subset \mathbb{Z}[t]$ is elementary weakly subintegral. Motivated by such examples we introduce the following definition:

Definition 5. A monomial subring A of $B = \mathbb{Z}[t]$ is a subring generated as an abelian group by "monomials" at^i $(a \in \mathbb{Z},\ i \geq 1)$.

As an abelian group $A = \mathbb{Z} \oplus (\oplus_{i \geq 1} \mathbb{Z} b_i t^i)$ for integers $b_i \geq 0$. For simplicity assume that $t^d \in A$ for some $d \geq 1$. Without loss of generality we may assume that $b_i \neq 0$ for i sufficiently large. It is convenient to describe A by giving its *coefficient sequence* $1, b_1, b_2, \ldots$.

Easy Observations 6.

1. If c_i is an arbitrary sequence of non-negative integers (with $c_0 = 1$) then $\mathbb{Z} \oplus (\oplus_{i \geq 1} \mathbb{Z} c_i t^i)$ is a ring if and only if $c_{i+j} \mid c_i c_j$.

2. The above condition can be checked at each prime, so that if q is a prime and $c^{[q]}$ is the q-primary part of an integer c then $C = \mathbb{Z} \oplus (\oplus_{i \geq 1} \mathbb{Z} c_i t^i)$ is a ring if and only if $C^{[q]} = \mathbb{Z} \oplus (\oplus_{i \geq 1} \mathbb{Z} c_i^{[q]} t^i)$ is a ring for all primes q.

3. The sequence $\{b_i\}$ becomes periodic if i is sufficiently large. Explicitly there are integers N and δ such that if $i \geq N$ then $b_{i+\delta} = b_i$. (Take δ to be the smallest such period.)

4. If we obtain the sequence a_i by extending b_i back to $i = 0$ periodically (i.e. $a_i = b_{i+n\delta}$ for $n >> 0$) then $\mathbb{Z} \oplus (\oplus_{i \geq 1} \mathbb{Z} a_i t^i)$ is a ring (which we will call the *stabilization* of A). The stabilization of A is a monomial subring of $\mathbb{Z}[t]$ which is larger that A.

5. If we let $\overline{a}$ denote the "squarefree" part of an integer A then $\overline{A} = \mathbb{Z} \oplus (\oplus_{i \geq 1} \mathbb{Z} \overline{a}_i t^i)$ is a ring.

Theorem 7. If A is a monomial subring of $B = \mathbb{Z}[t]$ and $M = at^i$ then
(a) $A[M]$ is subintegral over A if and only if $M^i \in A$ for $i >> 0$.
(b) $A[M]$ is weakly subintegral over A if and only if there is an integer valued polynomial $f \in \mathbb{Q}$ such that $f(0) = 1$ and $f(i)M^i \in A$ for $i >> 0$.

Our goal is to determine the seminormal and weakly normal monomial subrings of $\mathbb{Z}[t]$ (which contain t^δ for some integer $\delta \geq 1$). From the above observations it follows that A is seminormal (respectively weakly normal) in $\mathbb{Z}[t]$ if and only if $A^{[q]}$ is seminormal (respectively weakly normal) in $\mathbb{Z}[t]$. Thus it suffices to consider the case where all the coefficients are powers of one prime q. Then we have

Theorem 8. Let A be a monomial subring of $\mathbb{Z}[t]$ and suppose that all the coefficients of A are powers of one prime q. Then
(a) A is seminormal in $\mathbb{Z}[t]$ if and only if the coefficient sequence of A is of the form $1, q, q, \ldots,\ q, 1, \ldots$ with some period $\delta \geq 1$.
(b) A is weakly normal in $\mathbb{Z}[t]$ if and only if the coefficient sequence of A is $1, q, q, \ldots, q, 1, \ldots$ with some period $\delta \geq 1$, $\mathrm{GCD}(\delta, q) = 1$.

Theorem 9. Let $A = \mathbb{Z} \oplus (\oplus_{i \geq 1} \mathbb{Z} b_i t^i)$ be an arbitrary monomial subring of $B = \mathbb{Z}[t]$. Then ${}^{+}_{B}A = \overline{C}$, where C is the stabilization of A, and ${}^{*}_{B}A$ is obtained by first forming $D = {}^{+}_{B}A$ and then for each prime q replacing $D^{[q]}$ (with coefficient sequence $1, q, \ldots, q, 1, \ldots$ and period δ_q) by the corresponding ring with period $\delta_q / \delta_q^{[q]}$.

Examples 10. (a) Let $A = \mathbb{Z}[6t, 3t^2, 2t^3, 3t^4, t^6]$ with coefficient sequence $1, 6, 3, 2, 3, 6, 1 \ldots$ of period 6. Then $A^{[2]}$ has coefficient sequence $1, 2, 1 \ldots$ repeating with period 2 and $A^{[3]}$ has coefficient sequence $1, 3, 3, 1 \ldots$ of period 3 (i.e. $A^{[2]} = \mathbb{Z}[2t, t^2]$ and $A^{[3]} = \mathbb{Z}[3t, 3t^2, t^3]$). Both $A^{[2]}$ and $A^{[3]}$ are seminormal so A is seminormal in $\mathbb{Z}[t]$. However ${}^{*}_{B}A = \mathbb{Z}[t]$.
(b) Let $A = \mathbb{Z}[12t, 6t^2, t^3]$. Then A is already stable and $D = {}^{+}_{B}A = \mathbb{Z}[6t, 6t^2, t^3]$. Here $D^{[2]}$ has coefficient sequence 1 2 2 1 ... (with period 3). $D^{[2]}$ is already weakly normal in $\mathbb{Z}[t]$. However $D^{[3]}$ has coefficient sequence 1 3 3 1 ... (with period 3). This is weakly subintegral in $\mathbb{Z}[t]$ so ${}^{*}_{B}A = \mathbb{Z}[2t, 2t^2, t^3]$.
(c) $A = \mathbb{Z}[6t, 2t^2, 3t^3, 2t^4, t^6]$ (with coefficient sequence $1, 6, 2, 3, 2, 6, 1 \ldots$) is weakly normal in $\mathbb{Z}[t]$.
(d) $A = \mathbb{Z}[t^3, 3t^4, 6t^5]$ has coefficient sequence $1, 0, 0, 1, 3, 6, 1, 3, 3, 1, \ldots$ (now periodic of period 3). This stabilizes to 1 3 3 1 repeating with period 3, so ${}^{+}_{B}A = \mathbb{Z}[3t, 3t^2, t^3]$ and ${}^{*}_{B}A = \mathbb{Z}[t]$.

Characterization of weak subintegrality.

For any commutative ring R define a multiplication of $R[[T]]$ by $T^i * T^j = \binom{i+j}{i} T^{i+j}$. It $r \in R$ define $\mathrm{Exp}(r)$ by $\mathrm{Exp}(r) = 1 + \sum_{i=1}^{\infty} r^i T^i$. Then
$\mathrm{Exp}(r+s) = \mathrm{Exp}(r) * \mathrm{Exp}(s)$.

Theorem 11. Let $A \subseteq B$ be an arbitrary extension of commutative rings and let $b \in B$. Then $b \in {}^*_B A$ if and only if there is a polynomial $f \in B[T]$ with $f(0) = 1$ such that $f(T) * \mathrm{Exp}(r) \in A[[T]]$.

Problem 12. Is there a similar characterization of subintegrality, in the case when A does not contain $\mathbb{Q}$?

Example 13. Suppose that $b^i \in A$ for $i \geq n$. Let $f(T) = 1 + \sum_{i=1}^{n-1} (-1)^i b^i T^i$. Then $f(T) * \mathrm{Exp}(b) = 1+$ terms in $b^i, i \geq n$ so $f(T) * \mathrm{Exp}(b) \in A[[T]]$.
Explicitly $f(T) * \mathrm{Exp}(b) = 1 + \sum_{i=n}^{\infty} (-1)^{n-1} \binom{i-1}{n-1} b^i T^i$.
Thus in order to have $f(T) * \mathrm{Exp}(b) \in A[[T]]$ (and hence that $b \in {}^*_A B$) we require only that $\binom{i-1}{n-1} b^i \in A$.
For example take $A = \mathbb{Z}[\{\binom{i-1}{n-1} t^i\}_{i \geq n}]$ and $B = \mathbb{Z}[t]$. In this case $A \subseteq B$ is weakly subintegral.

Example 14. Let $n = 3$. Then $A = \mathbb{Z}[t^3, 3t^4, 6t^5, 10t^6 \ldots] = \mathbb{Z}[t^3, 3t^4, 6t^5]$ which has coefficient sequence $1, 0, 0, 1, 3, 6, 1, 3, 3, 1, \ldots$ and stabilization sequence $1, 3, 3, 1, \ldots$, corresponding to $\mathbb{Z}[3t, 3t^2, t^3]$.

We wish to determine the stabilization sequence of $A = \mathbb{Z}[\{\binom{i-1}{n-1}t^i\}_{i\geq n}]$ for all n. First a preliminary theorem:

Theorem 15. Let n and i be a fixed integers, $n \geq 1$ and $1 \leq i \leq n$. Then $\{\mathrm{GCD}\{\binom{d-1}{n-1}\} \mid d \equiv i \text{ MOD } n\} = n/\mathrm{GCD}(n, i)$.

Examples 16. If $n = 3$ then the sequence $\{n/\mathrm{GCD}(n, i)\}$ (as i varies) is the sequence $1, 3, 3, 1, 3, 3, 1, \dots$. If $n = 6$ then $\{n/\mathrm{GCD}(n, i)\}$ is the sequence $1, 6, 3, 2, 3, 6, 1, \dots$.

It follows from the Theorem and a not too difficult calculation that the stabilization of $A = \mathbb{Z}[\{\binom{i-1}{n-1}t^i\}_{i\geq n}]$ is $\mathbb{Z}[\{(n/\mathrm{GCD}(i, n))t^i\}_{1\leq i\leq n}]$ which has stabilization sequence $\{n/\mathrm{GCD}(i, n)\}_{i\geq 0}$.

Exercise 17. Give an example of an extension $A \subseteq B$ that satisfies 1(ii) but not 1(i).

Most of the above material is found in [1] (available on my web page). Basic references are [3] for subintegrality and [4] for weak subintegrality. Theorem 11 is found in [2].

References

[1] Les Reid and Leslie G. Roberts. Monomial subrings and systems of subintegrality. To appear in *JPAA*.

[2] Les Reid, Leslie G. Roberts, and Balwant Singh. On weak subintegrality. *JPAA*, 114:93–109, 1996.

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[4] H. Yanagihara. On an intrinsic definition of weakly normal rings. *Kobe J. Math.*, 2:89–98, 1985.