Projective Monomial Curves

$$\mathscr{S} = \{a_1, \ldots, a_n\}$$

$$0 < a_1 < a_2 < \cdots < a_n = d, \ \gcd(\{a_i\}) = 1.$$

Let S be the affine semigroup generated by

$$\{(a_d, 0), (d - a_1, a_1), \ldots (d - a_{n-1}, a_{n-1}), (0, d)\}.$$

- $R = K[S] \cong k[s^d, s^{d-a_1}t^{a_1}, \ldots, s^{d-a_{n-1}}t^{a_{n-1}}, t^d]$
- $C = \mathrm{Proj}(R)$ is a projective monomial curve embedded in $\mathbb{P}^n_K$ with homogeneous coordinate ring R.
- $\{s^d, t^d\}$ is a system of parameters for R, and R is Cohen-Macaulay if and only if $\{s^d, t^d\}$ is a regular sequence on R if and only if t^d is a non-zero-divisor in R/s^dR.

Notation: We will informally refer to $\mathscr{S}$ as the curve, will say that $\mathscr{S}$ or C is Cohen-Macaulay if R is Cohen-Macaulay.

The ring R is $\mathbb{N}$-graded as follows: if $s^a t^b \in R$ then $d \mid (a+b)$ and we define $\deg(s^a t^b) = (a+b)/d$. Then $R = \oplus_{i \geq 0} R_i$ is a standard graded K-algebra of dimension 2 and we define

- The Hilbert function $H_R(t) = \dim_K R_t, t \geq 0$
- The h-vector of R is the second difference sequence $\nabla^2 H_R$.

Easy Observation: If R is Cohen-Macaulay than $\nabla^2 H_R$ is the Hilbert function of $R/(s^d, t^d)R$, so $\nabla^2 H_R$ has no negative entries.

Motivating Problem: Is the converse true, i.e. if $\nabla^2 H_R$ has no negative entries, is R Cohen-Macaulay?

Many simple examples suggest that the converse might be true. For example, the first non-Cohen-Macaulay curve in many expositions is $\mathscr{S} = \{1, 3, 4\}$, which has h-vector 1 2 2 -1 0 $\rightarrow$.

Answers:

- Yes in $\mathbb{P}^2_K$. Every curve is a hypersurface, hence a complete intersection, hence Cohen-Macaulay.
- Yes in $\mathbb{P}^3_K$. Harder.
- No in $\mathbb{P}^i_K$, $i \geq 4$.

Example: $\mathscr{S} = \{5, 9, 11, 20\}$, so that $K[S] = K[s^{20}, s^{15}t^5, s^{11}t^9, s^9t^{11}, t^{20}]$. Then $\nabla^2 H_R$ is 1 3 5 6 5 0 $\rightarrow$, which has no negative entries. However $s^{56}t^{24} = (s^{15}t^5)(s^{15}t^5)(s^{15}t^5)(s^{11}t^9) \in K[S]$ but $(36, 24) \notin S$ so $s^{56}t^{24}$ represents a non-zero element of $K[S]/s^{20}K[S]$. But $(56, 44) = (20, 0) + 4(9, 11) \in S$ so $t^{20}(s^{56}t^{24}) = 0$ in $K[S]/s^{20}K[S]$. Therefore t^{20} is a zero-divisor in $K[S]/s^{20}K[S]$, and $\mathscr{S} = \{5, 9, 11, 20\}$ is not Cohen-Macaulay.

Some Data: I have checked all projective monomial curves up to degree 20. In the table c_d denotes the number of curves of degree d, ncm_d denotes the number that are not Cohen-Macaulay, and e_d denotes the number that are not Cohen-Macaulay and have no negative entry in their h-vector. The source column says where the e_d curves lie, for example $(10, 5)$ means 10 curves in $\mathbb{P}^5$.

For a curve $\mathscr{S}$ of degree d the integers 1 to $d-1$ might or might not be in $\mathscr{S}$ so, omitting cases where $\gcd(\mathscr{S}) \neq 1$, c_d is approximately 2^{d-1}. If d is prime then $c_d = 2^{d-1} - 1$, otherwise it will be a bit smaller.

Comments on the data:

- Most curves are not Cohen-Macaulay. Les Reid and I have been able to prove that the fraction that are not Cohen-Macaulay goes to 1 as $d \to \infty$.

- However there are still lots of Cohen-Macaulay curves. We have been able to prove that as $d \to \infty$ the number of Cohen-Macaulay curves grows exponentially, as least as $2^{d/2}$, although we do not know the exact rate.

- The number that are not Cohen-Macaulay and have no negative entry in their h-vector seems to be rather small, although I do not know how to prove this.

- An example in degree 12 is $\mathscr{S} = \{4, 7, 10, 11, 12\}$. The four cases of degree 20 in $\mathbb{P}^4$ are $\{(9, 11, 17, 20), (9, 11, 15, 20), (5, 9, 11, 20), (3, 9, 11, 20)\}$. Of the last four the second and third are isomorphic (under interchanging the coordinates of S) as are the first and last. Hence there are only two non-isomorphic examples of degree 20 in $\mathbb{P}^4$.

- It appears that $e_d \to \infty$ as $d \to \infty$ but I do not know how to prove this. However there are an infinite number of such curves.

Regularity: Non-Cohen-Macaulay curves with no negative entry in their h-vector have interesting Betti diagrams. For example, the Betti diagram for $K[S]$ in the case $\mathscr{S} = \{5, 9, 11, 20\}$ is

```
total: 1 6 10 6 1
    0: 1 .  . . .
    1: . 1  . .
    2: . 1  . . .
    3: . 1  1 . .
    4: . 3  9 5 1
    5: . .  . 1 .
```

The crucial feature here is the zero in the lower right corner (row 5, column 4).

- The number of the last non-zero row in the Betti diagram of $K[S]$ (regarded as a module over the coordinate ring of projective space) is the regularity of $K[S]$ (5 for $\mathscr{S} = \{5, 9, 11, 20\}$).

- For a curve in $\mathbb{P}^n$ the number of the last non-zero column is $n - 1$ if the curve is Cohen-Macaulay and n if not. (Auslander-Buchsbaum Theorem). E.g. in the above diagram for $\mathscr{S} = \{5, 9, 11, 20\}$ which is a non-Cohen-Macaulay curve in $\mathbb{P}^4$, the last non-zero column is the 4-th.

- We will say that regularity is attained in column i if the entry in the last row and column i is non-zero. E.g. for $\mathscr{S} = \{5, 9, 11, 20\}$ regularity is attained only in column 3.

Theorem: (Omidali) If the entry in the lower right corner of the Betti diagram of a non-Cohen-Macaulay curve in $\mathbb{P}^n$ is non-zero (equivalently, regularity is attained in column n) then the h-vector of the curve contains a negative entry.

Example: For $\mathscr{S} = \{1, 3, 4\}$, which is a non-Cohen-Macaulay curve in $\mathbb{P}^3$, the Betti diagram is

```
total: 1 4 4 1
    0: 1 . . .
    1: . 1 . .
    2: . 3 4 1
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and the h-vector is 1 2 2 -1 $\to$ 0.

Let $\mathscr{N}$ be the set of non-Cohen-Macaulay projective monomial curves which have no negative entry in their h-vector, and let $\mathscr{R}$ be the non-Cohen-Macaulay projective monomial curves for which regularity is not attained in the last column. The Theorem says that $\mathscr{N} \subseteq \mathscr{R}$. The sets $\mathscr{N}$ and $\mathscr{R}$ are both empty in degrees less than 12. The sets $\mathscr{N}$ and $\mathscr{R}$ are equal in degrees 12 through 17, so a small amount of searching might lead one to think that they are equal.

However $\mathscr{R}$ contins 812 elements of degree $d = 18$, of which $812 - 749 = 63$ have a negative entry in their h-vector. All 63 lie in $\mathbb{P}^7$ and have h-vector 1 6 12 -1 0 $\rightarrow$. They have several different Betti diagrams, a typical one of which is

```
total: 1 17 86 171 165 81 19 2
    0: 1  .  .   .   .  .  . .
    1: .  9  5   .   .  .  . .
    2: .  8 81 171 165 81 18 2
    3: .  .  .   .   .  .  1 .
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which is obtained, for example, from $\mathscr{S} = \{1, 2, 3, 6, 10, 13, 18\}$. The crucial feature of this diagram (compared with that for $\mathscr{S} = \{5, 9, 11, 20\}$) is that the 2 in column 7 is greater than the 1 in column 6. The difference $1 - 2 = -1$ ends up being the -1 in the h-vector, just as $1 - 1 = 0$ for the diagram of $\{5, 9, 11, 20\}$.

Similarly $\mathscr{R}$ contains 715 elements of degree $d = 19$, of which $715 - 685 = 30$ have a negative entry in their h-vector. All have h-vector 1 6 13 -1 0 $\rightarrow$.

Computation: Let Γ be the numerical semigroup generated by $\mathscr{S}$, and let Θ_i be the set of all elements of Γ which can be written as a sum of i elements of $\mathscr{S}$ (repetition is allowed). Set $\mathfrak{M}_i := \Theta_i \setminus (\bigcup_{j<i} \Theta_j)$. The elements of $\mathfrak{M}_i$ are those elements of Γ which can be minimally expressed as the sum of i elements of $\mathscr{S}$. A K-basis of $K[S]/s^d K[S]$ in degree i is now $\{s^{id-a}t^a \mid a \in \mathfrak{M}_i\}$ and $\dim_K(K[S]/s^d K[S])_i = \#\mathfrak{M}_i$.

A variant of the Computation: As before let Γ be the numerical semigroup generated by $\mathscr{S}$, and let Θ_i be the set of all elements of Γ which can be written as a sum of i elements of $\mathscr{S}$ (repetition is allowed). Set $\mathfrak{M}_i := \Theta_i \setminus (\bigcup_{j>i} \Theta_j)$. We now have the following: $K[\Gamma] = \{t^i \mid i \in \Gamma\}$ gives an affine piece $D_+(s^d)$ of our projective monomial curve C and $\{t^d \mid d \in \mathfrak{M}_i\}$ is a K-basis of $\mathfrak{m}^i/\mathfrak{m}^{i+1}$, i.e. we are computing the associated graded ring A of the local ring of C at the point corresponding to the 'origin' of $K[\Gamma]$. A is a one dimensional graded ring that might or might not be Cohen-Macaulay. The difference sequence of H_A will be the analogue of the h-vector of C and we might ask if this h-vector has negative entries, equivalently can H_A ever have a decrease.

The next table presents some data. In this table, n_d is the number of Γ with largest minimal generator d, ncm_d is the number of these that are not Cohen-Macaulay, and r is the fraction that are not Cohen-Macaulay. Observations, in contrast to the projective case, are (at least, for low degrees)

- Most cases are Cohen-Macaulay
- In no case was ∇H_A observed to have a negative entry.

Based on this data one might be tempted to believe that H_A can never decrease. But this is not the case. This question has been studied quite a bit, one one of the earliest examples is from Judy Sally's book "numbers of generators of ideals in local rings":
$\mathscr{S} = \{30, 35, 42, 47, 148, 153, 157, 169, 181, 193\}$ for which H_A is 1, 9, -1, 7, 9, 5, 0 $\rightarrow$.