THE CHOWLA PROBLEM AND ITS GENERALIZATIONS

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Primes

Infinitude of primes - Euclid in 300 B.C.

Infinitude of primes - Euler's product formula in 1737: For any real $x > 1$,

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^x}\right) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Infinitude of primes in A.P. - Conjectured by Legendre and proved by Dirichlet in 1840, the theorem states that

Theorem

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2 / 17
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Dirichlet characters

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For example, the trivial homomorphism from $(\mathbb{Z}/q\mathbb{Z})^*$ to $\mathbb{C}^*$ gives rise to the trivial character, namely,

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Remark

*The Dirichlet characters, \( \chi : \mathbb{N} \rightarrow \mathbb{C} \) are periodic functions with period \( q \).*
Let $\chi$ be a Dirichlet character of modulus $q$. Motivated by Euler, Dirichlet defined the series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

which is absolutely convergent for $\Re(s) > 1$.
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analogous to Euler's product formula. For example,

$$L(s, \chi_0) = \prod_{p \text{ prime}} \left(1 - \frac{\chi_0(p)}{p^s}\right)^{-1} = \prod_{p \text{ prime, } (p,q)=1} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$= \zeta(s) \prod_{p \text{ prime, } p|q} \left(1 - \frac{1}{p^s}\right),$$

where $\zeta(s)$ denotes the Riemann zeta function.
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Since there are $\phi(q)$ many Dirichlet characters with modulus $q$, Dirichlet proved that for any integer $a$ with $(a, q) = 1$,

$$\left[ \prod_{\text{prime } p, \ p \equiv a \mod q} \left( 1 - \frac{1}{p^s} \right)^{-1} \right]^{-\phi(q)} = \prod_{\chi \mod q} L(s, \chi)^{\overline{\chi(a)}},$$

for all $s \in \mathbb{C}$. 
Thus, we have

\[
\left[ \prod_{\substack{p \text{ prime,} \\ p \equiv a \mod q}} \left(1 - \frac{1}{p^s}\right)^{-1} \right]^{-\phi(q)} = \zeta(s) \left[ \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right) \right] \prod_{\chi \neq \chi_0} L(s, \chi)^{\chi(a)},
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Taking the limit at \( s \to 1^+ \) in the previous equation would lead to the result, unless the pole of \( \zeta(s) \) at \( s = 1 \) gets cancelled by a zero of \( L(s, \chi) \) at \( s = 1 \)!
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Thus, we have that if

\[ L(1, \chi) \neq 0, \text{ for all non-trivial characters with moduli } q, \]

then there are infinitely many primes in the arithmetic progressions \( a \mod q \) with \( a \) and \( q \) relatively prime.
Thus, we have

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This is the most difficult step in the proof for which Dirichlet discovered the class number formula!
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Then is it true that the series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0?$$
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More generally, one can ask the following: Fix a positive integer $q$. Let $f$ be an algebraic-valued (i.e., $\mathbb{Q}$-valued) arithmetic function, periodic with period $q$ and not identically zero. Then is it true that

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**Remark**

This question aims to generalize Dirichlet’s theorem that

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \neq 0,$$

for a non-trivial Dirichlet character $\chi$. 
Let $f$ be an arithmetic function, periodic with period $q$. Following Dirichlet, it is useful to define an $L$-series attached to $f$, namely,

$$L(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

which converges absolutely for $\Re(s) > 1$. 

Using the periodicity of $f$, we can rewrite $L(s, f)$ as follows:

$$\sum_{n=1}^{\infty} f(n) \frac{n}{n^s} = \frac{1}{q} \sum_{a=1}^{q} f(a) \sum_{k=0}^{\infty} \frac{1}{(a+qk)^s} = \frac{1}{q} \sum_{a=1}^{q} f(a) \sum_{k=0}^{\infty} \frac{1}{(k + a/q)^s}. $$

One observes that the inner summation is $\zeta(s, a/q)$, where $\zeta(s, x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$, for $\Re(s) > 1$ and $0 < x \leq 1$ is the Hurwitz zeta function.
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Thus, we have

$$L(s, f) = \frac{1}{q^s} \sum_{a=1}^{\infty} f(a) \zeta \left(s, \frac{a}{q}\right),$$

for $\Re(s) > 1$. 

In 1882, Hurwitz proved that $\zeta(s, x)$ obtains analytic continuation to the entire complex plane except for a simple pole at $s = 1$ with residue $1$.

Thus, Hurwitz's theorem implies that $L(s, f)$ extends to an analytic function on the entire complex plane except for a simple pole at $s = 1$ with residue $1/q \left(\sum_{a=1}^{\infty} f(a)\right)$. Hence, we have

$$\infty \sum_{n=1}^{\infty} f(n) n < \infty \iff q \sum_{a=1}^{\infty} f(a) = 0.$$

Thus, we have

$$L(s, f) = \frac{1}{q^s} \sum_{a=1}^{q} f(a)\zeta\left(s, \frac{a}{q}\right),$$

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$$\sum_{n=1}^{\infty} \frac{f(n)}{n} < \infty \iff \sum_{a=1}^{q} f(a) = 0.$$
Chowla (1964) [3] - (Following an argument outlined by Siegel)

i. $\mathbb{Q}$-valued,
ii. periodic with prime period $p$,
iii. $f(p) = 0$,
iv. odd, i.e., $f(p - n) = -f(n)$,
The Chowla Problem: History I

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- **Baker, Birch, Wirsing (1973) [1]** -
  
  i. \(\overline{\mathbb{Q}}\)-valued,
  
  ii. periodic with period \(q\),
  
  iii. \(f(r) = 0\) for all \(r\) such that \(1 < (r, q) < q\),
  
  iv. \(\mathbb{Q}(f(1), \cdots, f(q)) \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}\).
Baker, Birch, Wirsing (1973) [1] -

Constructed a basis for the $\mathbb{Q}$- vector space of odd, $\mathbb{Q}$-valued arithmetical functions $f$, periodic with period $q$ such that $L(1, f) = 0$. 

T. Chatterjee, R. Murty (2014) [2] - For a $\mathbb{Q}$-valued arithmetical function $f$, periodic with period $q$, if

\begin{align*}
    f_o(x) &:= f(x) - f(-x), \\
    f_e(x) &:= f(x) + f(-x),
\end{align*}

(thus, $f = f_o + f_e$, where $f_o$ is an odd function and $f_e$ is an even function) then $L(1, f) = 0 \iff L(1, f_o) = 0$ and $L(1, f_e) = 0$. 

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(thus, $f = f_o + f_e$, where $f_o$ is an odd function and $f_e$ is an even function) then

$$L(1, f) = 0 \iff L(1, f_o) = 0 \text{ and } L(1, f_e) = 0.$$
Given a function $f$ which is periodic with period $q$, define the Fourier transform of $f$ as

$$\hat{f}(x) := \frac{1}{q} \sum_{a=1}^{q} f(a) \zeta_q^{-ax},$$

where $\zeta_q = e^{2\pi i / q}$. 
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This can be inverted using the identity

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Thus, the condition for convergence of

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}$$

seen earlier, i.e., $\sum_{a=1}^{q} f(a) = 0$ can be interpreted as $\hat{f}(q) = 0$. 
We define the following functions that act as building blocks for even \( \overline{\mathbb{Q}} \)-valued arithmetical functions \( f \), periodic with period \( q \) and \( L(1, f) = 0 \).
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For a divisor \( d \) of \( q \) such that \( 1 < d < q \) and \( 1 \leq c \leq d - 1 \), define:

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F_{d,c} := F_{d,c}^{(1)} - F_{d,c}^{(2)},
\]

where,
Building blocks for even functions with $L(1, f) = 0$

We define the following functions that act as building blocks for even $\overline{\mathbb{Q}}$-valued arithmetical functions $f$, periodic with period $q$ and $L(1, f) = 0$.

For a divisor $d$ of $q$ such that $1 < d < q$ and $1 \leq c \leq d - 1$, define:

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where,

\[ F_{d,c}^{(1)}(m) = \begin{cases} 1/2 & \text{if } m \equiv c \mod q, \\ 0 & \text{otherwise}. \end{cases} \]

\[ F_{d,c}^{(2)}(m) = \begin{cases} 1/2 & \text{if } m \equiv \left(\frac{q}{d}\right)c \mod q, \\ 0 & \text{otherwise}. \end{cases} \]
Characterization of even functions $f$ such that $L(1, f) = 0$

(Joint work with T. Chatterjee and R. Murty.)

**Theorem**

Let $f$ be an even $\mathbb{Q}$-valued arithmetical function, periodic with period $q$ and

$$
\sum_{a=1}^{q} f(a) = 0.
$$

If $L(1, f) = 0$, then $f$ is an algebraic linear combination of the functions

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\{ \widehat{F}_{d,c} \mid d \text{ is a divisor of } q, 1 < d < q, 1 \leq c \leq d - 1 \}.
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Here, $\widehat{F}_{d,c}$ denotes the Fourier transform of $F_{d,c}$. 
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**Theorem**

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This theorem gives a necessary condition for $L(1, f)$ to vanish.
Conclusion

As a corollary of this theorem, we obtain that:

**Corollary**

Let $f$ be an algebraic-valued arithmetic function, periodic with prime period $p$. If $f$ is an even function (i.e, $f(p - n) = f(n)$), then

\[ L(1, f) \neq 0. \]
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Recall that Chowla proved non-vanishing of $L(1, f)$ when $f$ is $\mathbb{Q}$-valued and odd. Thus, the above corollary together with the Chatterjee-Murty theorem answers Chowla’s original question in the affirmative.
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Recall that Chowla proved non-vanishing of $L(1, f)$ when $f$ is $\mathbb{Q}$-valued and odd. Thus, the above corollary together with the Chatterjee-Murty theorem answers Chowla’s original question in the affirmative. Indeed, if $f$ is a $\mathbb{Q}$-valued arithmetic function, periodic with prime period $p$, then

$$L(1, f) \neq 0.$$
Thank you!

