Chapter 10: Compact Metric Spaces

10.1 Definition. A collection of open sets $\{U_i : i \in I\}$ in X is an *open cover* of $Y \subset X$ if $Y \subset \cup_{i \in I} U_i$. A *subcover* of $\{U_i : i \in I\}$ is a subcollection $\{U_j : j \in J\}$ for some $J \subset I$ that still covers Y. It is a *finite subcover* if J is finite.

10.2 Definition.

1. A metric space X is *compact* if every open cover of X has a finite subcover.

2. A metric space X is *sequentially compact* if every sequence of points in X has a convergent subsequence converging to a point in X.

10.3 Examples.

1. $(0, 1]$ is not sequentially compact (using the Heine-Borel theorem) and not compact. To show that $(0, 1]$ is not compact, it is sufficient find an open cover of $(0, 1]$ that has no finite subcover. But a moment's consideration of the cover consisting exactly of the sets $U_n := (\frac{1}{n}, 2)$ shows that this is just such a cover. That is, $\cup_{n \in \mathbb{N}} U_n = (0, 2) \supset (0, 1]$. But if F is any finite subset of $\{U_n : n \in \mathbb{N}\}$, then F contains an element U_k such that $k \geq i$ for each $U_i \in F$. But this means that $U_k \supseteq U_i$ for each $U_i \in F$ and hence $\cup F = U_k = (\frac{1}{k}, 2) \subsetneq (0, 1]$.

2. $[0, 1]$ is sequentially compact (applying Heine-Borel). In fact, $[0, 1]$ is also compact (as we will see shortly).

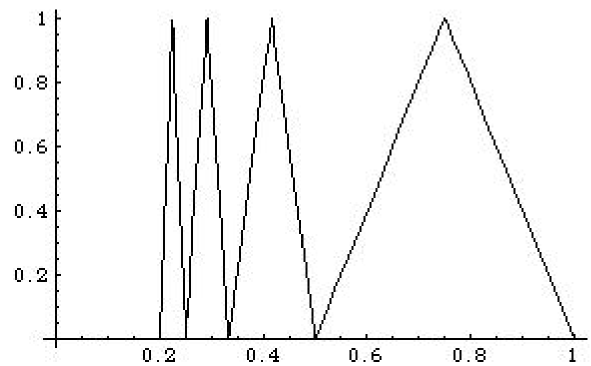
3. $\mathbb{R}$ is neither compact nor sequentially compact. That it is not sequentially compact follows from the fact that $\mathbb{R}$ is unbounded and Heine-Borel. To see that it is not compact, simply notice that the open cover consisting exactly of the sets $U_n = (-n, n)$ can have no finite subcover. Using reasoning similar to that of example 1, if F is a finite subset of $\{U_n : n \in \mathbb{N}\}$ then F contains an element U_k such that $k \geq i$ for each $U_i \in F$. But then $\cup F = U_k = (-k, k) \subsetneq \mathbb{R}$, so F cannot be an open cover for $\mathbb{R}$.

4. $X = C[0, 1]$ is not compact. Denote by $B_r(f)$ the open ball of radius r under the sup-norm centered at the function $f \in C[0, 1]$, and consider

the set $U = \{B_{\frac{1}{4}}(f) : f \in C[0,1]\}$. Clearly, U is an open cover of $C[0,1]$. Now define the sequence $(f_n)_{n\in\mathbb{N}}$ as

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{n+1} \\ 2n(n+1)(x - \frac{1}{n+1}) & \text{if } \frac{1}{n+1} < x \le \frac{2n+1}{2n(n+1)} \\ -2n(n+1)(x - \frac{1}{n}) & \text{if } \frac{2n+1}{2n(n+1)} < x \le \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < x \le 1 \end{cases}$$

The graphs of f_1, f_2, f_3, and f_4 are plotted below.



Notice that $\| f_m - f_n \|_\infty = 1$ whenever $m \neq n$. Hence, each element of U can contain at most one function from $(f_n)_{n\in\mathbb{N}}$, and therefore every finite subset of U fails to cover $C[0,1]$.

5. (Cantor's Intersection Theorem.) *If $C_1 \supset C_2 \supset C_3 \ldots$ is a decreasing sequence of nonempty sequentially compact subsets of $\mathbb{R}^n$, then $\cap_{k\ge1} C_k$ is non-empty.*
   To see this, choose the sequence $(a_n)_{n\in\mathbb{N}}$ so that $a_n \in C_n$ for every n. Clearly, (a_n) is a sequence in C_1. The compactness of C_1 tells us that (a_n) has a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$. Say that $a_{n_k} \to a$ as $k \to \infty$. Then a is the limit of a sequence in C_i for each i, which means that $a \in C_i$ for each i, and the result follows.

10.4 Definition. A metric space X is *totally bounded* if, for every $\epsilon > 0$, there exist $x_1, x_2, \ldots, x_k \in X$, with k finite, so that $\{B_\epsilon(x_i) : 1 \le i \le k\}$ is an open cover of X.

10.5 Examples.

1. Example 4 of 10.3 shows that the closed unit ball in $C[0,1]$ is not totally bounded.

2. $(0,1]$ is totally bounded since for any $\epsilon > 0$, $\{(i\epsilon, (i+1)\epsilon) : 0 \leq i \leq \frac{1}{\epsilon}, i \in \mathbb{Z}\}$ is an open cover. However, $(0,1]$ is not compact. (We will see shortly that the ingredient missing from $(0,1]$ and essential for compactness is in fact completeness.)

10.6 Definition. A collection of closed sets $\{C_i : i \in I\}$ has the *finite intersection property* if every finite subcollection has nonempty intersection.

10.7 Theorem. (The Borel-Lebesgue Theorem.) For a metric space (X, ρ), the following are equivalent:

1. X is compact.

2. Every collection of closed subsets of X with the finite intersection property has non-empty intersection.

3. X is sequentially compact.

4. X is complete and totally bounded.

Proof. We'll first show (1) implies (2). Consider a collection of closed subsets $\{C_i : i \in I\}$ of X having the finite intersection property, and assume that $\cap_{i \in I} C_i = \varnothing$. Put $U_i := C_i^c$ for each i, and notice that each U_i is open. We have $\cup_{i \in I} U_i = \cup_{i \in I} C_i^c = (\cap_{i \in I} C_i)^c = X$, so that $\{U_i : i \in I\}$ is an open cover of X. Since X is compact, there exists a finite subcover $\{U_{n_1}, U_{n_2}, \ldots, U_{n_k}\}$ of X. Hence,

$$\begin{aligned} X &= U_{n_1} \cup U_{n_2} \cup \ldots \cup U_{n_k} \\ &= (U_{n_1}^c \cap U_{n_2}^c \cap \ldots \cap U_{n_k}^c)^c \\ &= (C_{n_1} \cap C_{n_2} \cap \ldots \cap C_{n_k})^c, \end{aligned}$$

which means that $C_{n_1} \cap C_{n_2} \cap \ldots \cap C_{n_k} = \varnothing$, in contradiction with the finite intersection property.

The argument from (2) to (1) runs as follows: Suppose that $\{U_i : i \in$

$I\}$ is an open cover of X and put $C_i := U_i^c$ for each i. Suppose further that no finite subset of $\{U_i : i \in I\}$ covers X. Then if a subcollection $\{C_{n_1}, C_{n_2}, \ldots, C_{n_k}\}$ of $\{C_i : i \in I\}$ satisfies $C_{n_1} \cap C_{n_2} \cap \ldots \cap C_{n_k} = \varnothing$, we would have

$$\begin{aligned} & U_{n_1} \cup U_{n_2} \cup \ldots \cup U_{n_k} \\ &= (U_{n_1}^c \cap U_{n_2}^c \cap \ldots \cap U_{n_k}^c)^c \\ &= (C_{n_1} \cap C_{n_2} \cap \ldots \cap C_{n_k})^c \\ &= X, \end{aligned}$$

a contradiction with the assumption that no finite subset of $\{U_i : i \in I\}$ covers X. Thus, $\{C_i : i \in I\}$ has the finite intersection property. By (2), $\cap_{i \in I} C_i \neq \varnothing$, so $\cup_{i \in I} U_i \neq X$, meaning that $\{U_i : i \in I\}$ is not an open cover for X.

We'll now show that (3) follows from (1). Assume we have a sequence $(x_n)_{n \in \mathbb{N}}$ in X with no convergent subsequence. Since no term in the sequence can occur infinitely many times (otherwise we would have a convergent subsequence), we can assume without loss of generality that $x_i \neq x_j$ whenever $i \neq j$. Notice that each term of the sequence (x_n) is an isolated point of $\{x_n : n \in \mathbb{N}\}$, since otherwise, (x_n) would have a convergent subsequence. Hence, for each i there exists an open ball, call it U_i, centred at x_i with the property that $x_j \notin U_i$ for all $i \neq j$. Now put $U_0 := X \backslash \{x_n : n \in \mathbb{N}\}$. U_0 is open since its complement consists only of isolated points, and so is closed. Then $\{U_0\} \cup \{U_n : n \in \mathbb{N}\}$ is an open cover for X. But this open cover has no finite subcover, since any finite subcollection of these sets would fail to include infinitely many terms from the sequence (x_n) in its union.

To see that (3) implies (4), assume that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in a sequentially compact space X. Say that $(x_{n_k})_{k \in \mathbb{N}}$ is a convergent subsequence of (x_n) and that $x_{n_k} \to x$. Let $\epsilon > 0$ be given, and choose N so that $\rho(x_i, x_j) < \epsilon/2$ whenever $i, j \geq N$. Next, choose $n_k > N$ so that $\rho(x_{n_k}, x) < \epsilon/2$. Then we have

$$\rho(x, x_N) \leq \rho(x, x_{n_k}) + \rho(x_{n_k}, x_N) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus, $x_n \to x$ as $n \to \infty$, showing that X is complete. Next, we'll show that X is totally bounded.

Assume that X is not totally bounded, and take $\epsilon > 0$ such that X cannot be covered by a collection consisting of only finitely many ϵ-balls.

Choose $x_1 \in X$, $x_2 \in X \backslash B_\epsilon(x_1)$, then $x_3 \in X \backslash B_\epsilon(x_1) \backslash B_\epsilon(x_2)$, and so on. We thus have a sequence (x_n) which cannot contain a convergent subsequence since $\rho(x_i, x_j) \geq \epsilon$ for all $i \neq j$.

We can also obtain (3) from (4). Consider a sequence $(x_n)_{n\in\mathbb{N}}$ in X. Since X is totally bounded, we have, for every $n \in \mathbb{N}$, a finite set of points $\{y_1^{(n)}, y_2^{(n)}, \ldots, y_{r(n)}^{(n)}\}$ such that $X \subset B_{\frac{1}{n}}(y_1^{(n)}) \cup \ldots \cup B_{\frac{1}{n}}(y_{r(n)}^{(n)})$. Let $(S_n)_{n\in\mathbb{N}}$ be the sequence of finite subsets of X obtained by putting $S_n := \{y_1^{(n)}, \ldots, y_{r(n)}^{(n)})\}$. We can find a convergent subsequence $(z_n)_{n\in\mathbb{N}}$ of (x_n) using the following procedure: Since S_1 is finite, there is a $y_{n(1)}^{(1)} \in S_1$ such that $B_1(y_{n(1)}^{(1)})$ contains infinitely many points from (x_n). Select z_1 from $B_1(y_{n(1)}^{(1)})$. Now, since S_2 is finite, there is a $y_{n(2)}^{(2)} \in S_2$ such that $B_1(y_{n(1)}^{(1)}) \cap B_{\frac{1}{2}}(y_{n(2)}^{(2)})$ contains infinitely many points from (x_n). Choose z_2 from $B_1(y_{n(1)}^{(1)}) \cap B_{\frac{1}{2}}(y_{n(2)}^{(2)})$. Now continue this procedure for each $k > 1$, selecting $y_{n(k)}^{(k)}$ from S_k such that $\cap_{i=1}^k B_{\frac{1}{i}}(y_{n(i)}^{(i)})$ contains infinitely many points from (x_n), and then selecting z_k from $\cap_{i=1}^k B_{\frac{1}{i}}(y_{n(i)}^{(i)})$. (z_n) is clearly Cauchy, and by the completeness of X, (z_n) converges to a point in X.

Finally, we show that (3) implies (1). We'll need the following preliminary result: *If (X, ρ) is a sequentially compact metric space, and $\{U_i : i \in I\}$ an open cover for X. Then there is an $r > 0$ such that for each $x \in X$, $B_r(x) \subseteq U_i$ for some $i \in I$.* The proof is by contradiction. Assume that for some $r > 0$, there is an $x \in X$, possibly depending on r, such that for each $i \in I$, $B_r(x) \nsubseteq U_i$. Now choose the sequence $(x_n)_{n\in\mathbb{N}}$ in X so that $B_{\frac{1}{n}}(x_n) \nsubseteq U_i$ for all $i \in I$.

Since X is sequentially compact, (x_n) has a convergent subsequence, $(x_{n_k})_{k\in\mathbb{N}}$. Say that $x_{n_k} \to x$ as $k \to \infty$, where $x \in X$. There must be some i_0 such that $x \in U_{i_0}$, and, since U_{i_0} is open, an $r_0 > 0$ such that $B_{r_0}(x) \subseteq U_{i_0}$. So choose N such that $\rho(x, x_N) < \frac{1}{2}r_0$ and $\frac{1}{N} < \frac{1}{2}r_0$. Now if $y \in B_{\frac{1}{N}}(x_N)$, then

$$\begin{aligned}\rho(x,y) &\leq \rho(x,x_N) + \rho(x_N,y) \\ &< \frac{1}{2}r_0 + \frac{1}{2}r_0 \\ &< r_0,\end{aligned}$$

and hence $y \in B_{r_0}(x) \subseteq U_{i_0}$. It follows that $B_{\frac{1}{N}}(x_N) \subseteq B_{r_0}(x) \subseteq U_{i_0}$, a contradiction, and the preliminary result is proven.

Returning to the main argument, let $\{U_i : i \in I\}$ be an open cover of X. By the preliminary result, there exists an $r > 0$ such that for each $x \in X$, $B_r(x) \subset U_i$ for some $i \in I$. We know that (3) implies (4), thus X is totally bounded. That is, $X \subset B_r(y_1) \cup B_r(y_2) \cup \ldots \cup B_r(y_k)$ for some points $y_1, y_2, \ldots, y_k \in X$, with $k \in \mathbb{N}$. However, for each $i \in I$, we have that $B_r(y_i) \subset U_{j(i)}$ for some $j(i) \in I$. Thus $\{U_{j(1)}, U_{j(2)}, \ldots, U_{j(k)}\}$ is a finite subcover for X, and we're done. $\square$

10.8 Theorem. Let (X, ζ) and (Y, δ) be metric spaces and $f : X \to Y$ be a continuous function. Then for each compact subset $C \subset X$, $f(C) \subset Y$ is compact.

Proof. Let $\{U_i : i \in I\}$ be an open cover of $f(C)$, and for each $i \in I$, define V_i to be the pre-image of U_i under f. Notice that since f is continuous, each V_i is open. Thus, $\{V_i : i \in I\}$ is an open cover of C. But since C is compact, there exists a finite subcover $\{V_{i(1)}, V_{i(2)}, \ldots, V_{i(k)}\}$ for C, and hence $\{U_{i(1)}, U_{i(2)}, \ldots, U_{i(k)}\}$ is a finite subcover of $f(C)$. So, $f(C)$ is compact. $\square$