MATH 328: Chapter 4: Theorem of Arzela-Ascoli

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1 Definitions

In this chapter, we begin to discuss the ways in which $\mathbb{R}^n$ differs from $C[0,1]$. In particular, we compare the characterization of compact subsets of $\mathbb{R}^n$ by Heine-Borel with the characterization of compact subsets of $C[0,1]$ by Arzela-Ascoli. We find that subsets of $C[0,1]$ must satisfy more conditions than subsets of $\mathbb{R}^n$ if they are to be compact.

Before we can begin to investigate this, we have a few preliminary definitions to recall from point-set topology. In the following definitions, X is our normed vector space, A is a subset of the vector space.

Closed. *We call A a closed subset of X if, for any convergent sequence $(f_n)_{n\geq 1} \subset A$, the limit point is also in A.*

Open. *We call A an open subset if, $\forall\ x \in A$, $\exists\ \delta > 0$ such that $y \in X$, $\|y - x\| < \delta \Rightarrow y \in A$.*

Bounded. *We call A a bounded subset if $\exists M > 0$ such that, $\forall x \in A$, $\|x\| < M$.*

Compact. *We call A a compact subset if all sequences $(f_n)_{n\geq 1} \subset A$ have a convergent subsequence $(f_{n(k)})$ with limit point in A.*

2 Remarks

We recall that for a subset $A \subset X$, A is open if and only if A^C is closed.

Proof. A is open $\Rightarrow A^C$ is closed:
We consider a convergent sequence $(f_n)_{n\geq 1} \subset A^C$. Assume that the limit point, x, is not in A^C. Then, x must be in A. Since A is open, $\exists\ \delta > 0$ such that $B_\delta(x) \subset A$. However, since x is the limit of a convergent sequence, all but finitely many points in the sequence must be within δ of x. But all of those infinitely many points are then in A, which contradicts our initial statement that the entire sequence is in A^C. And so, the limit point must be in A^C for all such sequences, and so A^C is closed.
A^C is closed$\Rightarrow A$ is open :

We assume, to start, that A is not open. Then, $\exists x \in A$ such that, $\forall\ \delta > 0$, the set $B_\delta(x) \cap A^C$ is non-empty. We then define a sequence $(f_n)_{n\geq 1} \subset A^C$ as follows. Define δ_n to be $\frac{1}{2^n}$, and x_n to be any element in $B_{\delta_n}(x) \cap A^C$. This sequence converges to x. However, each element of the sequence is in A^C by definition, and x is in A. This contradicts the closedness of A^C, and so our assumption that A is not open must be false. □

We also note that $A \subset X$ compact implies that A is closed and bounded.

Proof. Let $(f_n)_{n\geq 1} \in A$ be a sequence in A which converges to f. Since $(f_n)_{n\geq 1}$ converges to f, so do all of its subsequences. Since A is compact, this implies that f is in A. And so, we see that all convergent sequences in A must converge to an element of A, which means exactly that A is closed. Similarly, if A is unbounded, there would be sequence $(f_n)_{n\geq 1} \in A$ with norm monotonely increasing and unbounded. Since the sequence defined by $\|f_n\|$ is monotonely increasing and unbounded, it cannot have convergent subsequences. We recall, however, that a necessary condition for convergence of a sequence of functions is that its norm converges. Since the sequence $\|f_n\|$ has no convergent subsequences, this implies that f_n has no convergent subsequences. And so we have shown that an unbounded set cannot be compact. □

We will soon see that for $\mathbb{R}^n$, the converse is true as well. That is, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. For $C[0,1]$, this is not true. There are subsets of $C[0,1]$ which are closed and bounded, but not compact.

Before investigating this, we will prove the Heine-Borel Theorem, which characterizes the compact subsets of $\mathbb{R}^n$.

3 Heine-Borel Theorem

Heine-Borel Theorem. *For any subset $A \in \mathbb{R}^n$, A is compact if and only if A is closed and bounded.*

Proof. A is compact $\Rightarrow$ A is closed and bounded: Proved in section 2.
A is closed and bounded $\Rightarrow$ A is compact:
Let $(x_n)_{n\geq 1}$ be a sequence in A. Since A is bounded, any sequence in A must also be bounded, and so $(x_n)_{n\geq 1}$ is a bounded sequence.
By the Bolzano-Weierstrass Theorem, this implies that there is a convergent subsequence of $(x_n)_{n\geq 1}$, and of course this sequence is also in A. Let this subsequence have limit point x. Since the subsequence is in A, and A is closed, its limit point must also be in A. And so we have found, for any sequence in A, a convergent subsequence with limit point in A. And so A is compact. □

4 Examples

We will now examine one of the simplest compact subset of $\mathbb{R}^n$, the unit ball. This is defined by:

$\overline{B_1(0)} = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$
This set is clearly closed and bounded, and so it is compact.

It is natural to consider the analogue of the unit ball in C[0,1], which we define by:
$\overline{B_1(0)} = \{f \in C[0,1] : \|f\|_\infty \leq 1\}$
This set is also closed and bounded, by inspection. However, we will show that it is not compact.

Consider the sequence defined by $f_n(x) = x^n$. It is clear that $||f_n||_\infty = 1$ for all n, and so this sequence is entirely contained in the unit ball.

Claim. *This sequence converges pointwise to the 0-function everywhere on [0,1] except at x = 1, where it converges to 1.*

Proof. Consider a point $a \in$ [0, 1). Let $\epsilon > 0$ be given. Then, for all $n > \frac{ln(\epsilon)}{ln(a)}$, we have:
$a^n < a^{\frac{ln(\epsilon)}{ln(a)}}$
$a^n < \epsilon$
And so, f_n converges to 0 pointwise everywhere on [0,1). However, $f_n(1) = 1$ for all n. And so the sequence converges pointwise to 0 on [0,1) and to 1 at $x = 1$. □

However, the limit function is clearly not continuous. And so, there is a sequence in the unit ball which converges pointwise to a function that is not even in C[0,1]. Since the sequence converges to this function, we recall that all subsequences must also converge pointwise to the same function. But we recall also that the uniform limit of a series of continuous function must itself be continuous, and that uniform convergence of a sequence to a function implies pointwise convergence of the sequence to the same function. This shows, then, that there is a sequence in B_1 for which no subsequence converges to a function in B_1. And so the unit ball in C[0,1] is not a compact set.
Thus, we have found a sequence in a closed and bounded subset of C[0,1] which has no subsequences that converge to an element of the subset. We can see that the Heine-Borel conditions for compactness are still necessary for subsets of C[0,1], but they are not sufficient.

Intuitively, this example shows that the unit ball in $\mathbb{R}^n$ is much "smaller" than the unit ball in C[0,1]. We will see in later sections that the difference can be understood in terms of the vector space structure of the two sets. $\mathbb{R}^n$ is a finite-dimensional vector space, and so bounded sequences intuitively only have a few directions to go in, and thus they must cluster up in at least one direction. In contrast, C[0,1] is infinite-dimensional, and so sequences, even in bounded sets, have an infinite number of "directions" in which they can go without clustering. This intuitive idea of size based on dimension will be formalized later. For now, we merely confirm that $\mathbb{R}^n$ is in fact finite-dimensional, while C[0,1] is infinite-dimensional.

To see that $\mathbb{R}^n$ is finite-dimensional, it is enough to find a finite spanning set. We define the n vectors e_i, i $\in \{1, 2, .., n\}$, to be 1 in their i'th entry and 0 in all other entries. We then note that any vector $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ may be written in the form $x = \sum_{i=1}^{n} x_i e_i$. This shows that these n vectors are a finite spanning set, and thus that there exists a finite basis. We recall that in fact this set is such a finite basis, which we usually call the standard basis.

To see that $C[0,1]$ is infinite-dimensional, it is enough to show an infinite set of vectors which are linearly independent. We define the family of vectors $\{x^n\}_{n\geq 1}$, which is certainly an infinite set of vectors in $C[0,1]$. To show that they are linearly independent, we assume that they are not. Then there are not all zero constants α_i and an integer m such that the following equality holds for all $x \in$ [0,1]:

$$\alpha_m x^m + \alpha_{m-1} x^{m-1} + .. + \alpha = 0 \tag{1}$$

But by the fundamental theorem of algebra, this polynomial can have at most m distinct real roots if it is not the 0-polynomial. And so for the equality to hold, it must be the 0-polynomial, showing that in fact the infinite set is linearly independent.

We see that the characterization of compact sets in $C[0,1]$ will thus be more difficult.

5 Motivation

We will see that the additional requirement for a subset of $C[0,1]$ to be convergent is somehow related to requiring all of the elements of the subset to be close to each other. To make this precise, we will shortly be introducing the definition of a property called equicontinuity, which is meant to deal with continuity of the entire set at once. Before this, we will motivate the idea of equicontinuity.
Consider a sequence $\{f_n\}_{n\geq 1}$ of $C[0,1]$ which converges uniformly to the continuous function f. For all n, we may then write, by the triangle inequality:

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f(x)| + |f(x) - f(y)| + |f_n(y) - f(y)| \tag{2}$$

We consider now a given $\epsilon > 0$. Since f_n is continuous for all n, there exists a $\delta_n > 0$ such that $|x - y| < \delta_n \Rightarrow |f_n(x) - f_n(y)| < \frac{\epsilon}{3}$. We also note that, since f_n converges to f, we may make the first and third terms in the inequality smaller than $\frac{\epsilon}{3}$ by requiring n to be sufficiently large, larger than some constant N. We note, from the definition of convergence, that this constant N is completely independent of x, y. Finally, in our preliminary work, we define $\delta_0 > 0$ to have the property that $|x - y| < \delta_0 \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{3}$. We know this is possible, because f is continuous.

We now define δ to be the maximum of all δ_n with $0 \leq n \leq N$. Since this is smaller than all δ_n's, we note immediately that, for all $n \leq N$, $|x - y| < \delta \Rightarrow |f_n(x) - f_n(y)|$. For $n > N$, however, we consider the above equation. Since $n > N$, the first and third terms are both less than $\frac{\epsilon}{3}$. Since $\delta < \delta_0$, the middle term is also less than $\frac{\epsilon}{3}$. And so we have again that $|x - y| < \delta \Rightarrow |f_n(x) - f_n(y)|$. And so we have found a δ for continuity arguments that is in fact independent of which function in the sequence we are looking at. It turns out that there are many sets of functions in C[0,1] in which it is not possible to find a $\delta > 0$ for all $\epsilon > 0$ which is independent of which element of the set is being dealt with.

Sets of functions for which such a δ can be found are called equicontinuous, and the above constitutes a formal proof that all convergent sequences of functions, viewed as sets, are in fact equicontinuous.

6 Equicontinuity

Equicontinuity. *Consider a subset $A \in C[0,1]$. We call A equicontinuous if,* $\forall \epsilon > 0$, $\exists \delta > 0$ *such that,* $\forall f \in A$, $\forall x, y \in [0,1]$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

This is very similar to the definition for continuity of a specific function f in A. The only difference is that, for a given ϵ, we must choose a $\delta > 0$ which works for all possible functions f in A. That is, we must choose the δ before we are allowed to look at the function, making this a property of a set rather than a function, whereas in proving continuity, we choose our δ for a specific function.

7 Remark

Since we are trying to show that equicontinuity is the additional property which compact sets in $C[0,1]$ contain, our example of a closed, bounded but not compact subset of $C[0,1]$ found in section 4, should fail to be equicontinuous. Fortunately, this is the case. We will prove that the unit ball in $C[0,1]$ is not equicontinous. Recall that the unit ball contained the sequence $\{x^n\}_{n\geq 1}$.

Proof. Let $\epsilon = \frac{1}{2}$, and assume that there exists such a $\delta_1 > 0$, so that the conditions for equicontinuity are satisfied. Define $\delta = \min\{\delta_1, 1\}$. Since this is at most δ_1, it must also work. Now consider $x = 1$, $y = 1$ - $\frac{\delta}{2}$. Clearly,
$|x - y| = |1 - 1 + \frac{\delta}{2}| = |\frac{\delta}{2}| < \delta$.
We are assuming that our set is equicontinuous, and so this implies that, $\forall f \in B_1(0)$, $|f(1) - f(1 - \frac{\delta}{2}| < \frac{1}{2}$. However, we have already seen that the sequence $\{x^n\}_{n\geq 1}$ is in the unit ball, and that it converges to 0 for all x $\in$ [0,1) and to 1 for $x = 1$. And so, $|f(1) - f(1 - \frac{\delta}{2})|$ can be made arbitrarily close to 1 for any fixed $\delta > 0$. And so the unit ball is not equicontinuous, although it is closed and bounded. □

We are now ready to prove the main result of this chapter, the Arzela-Ascoli Theorem.

8 Arzela-Ascoli Theorem

Arzela-Ascoli Theorem. *For $A \subset C[0,1]$, A is compact if and only if A is closed, bounded, and equicontinuous.*

Proof. A compact $\Rightarrow$ A closed, bounded, and equicontinuous.
We have already shown, in section 2, that A must be closed and bounded. It remains to show that A is equicontinuous.
To do so, we assume first that A is compact but not equicontinuous. Since it is not equicontinuous, we note that $\exists \epsilon > 0$ such that $\forall \delta\ \exists x, y \in [0,1]$ and $f \in A$ such that $|x-y| < \delta$, but $|f(x)-f(y)| > \epsilon$. In particular, we can create a chain of δ's, which we label by $\delta_n = \frac{1}{n}$, such that $\exists x_n, y_n \in [0,1]$ and $f_n \in A$ such that $|x_n - y_n| < \delta_n = \frac{1}{n}$, but $|f_n(x_n) - f_n(y_n)| > \epsilon$. This of course defines at least one sequence of functions in A. We choose one sequence of functions with this property, and we note that by the above this sequence cannot possibly be equicontinuous. Also, all subsequences $f_{n(k)}$ of this sequence would clearly also have the property that for the $n(k)$'th function in the sequence, $\exists x_{n(k)}, y_{n(k)} \in [0,1]$ such that $|x_{n(k)} - y_{n(k)}| < \delta_n$ but $|f_{n(k)}(x_{n(k)}) - f_{n(k)}(y_{n(k)})| > \epsilon$, by the above. And so,it is also true that no subsequence can be equicontinuous. However, we have shown already that all convergent sequences must in fact be equicontinuous. And so, under the assumption that A is not equicontinuous, we have demonstrated the existence of a sequence in A with no subsequence that converges. This is a contradiction with the assumption that A is compact, and so we conclude that A must be equicontinuous.

A closed, bounded, and equicontinuous $\Rightarrow$ A compact.
We begin by considering an arbitrary sequence $\{f_n\}_{n\geq 1}$ in A. We must show that it contains a convergent subsequence. Unfortunately, it is not very clear how to do this.
Intuitively, we may look at the interval [0,1], and find a subsequence which converges pointwise at one point, x_0. We could then find a subsequence of that subsequence which converged at a second point, x_1, and so on. This would work if [0,1] had only finitely many points. Unfortunately, the interval has uncountably many points, and so this strategy must be modified. The first modification is to use a diagonal argument, familiar from previous arguments in analysis, to extend convergence of a subsequence from a finite number of points to a countable set of points. We will then use the equicontinuity property to extend convergence at a well-chosen countable set of points to uniform convergence over the entire interval [0,1]. For now, we continue the proof.

We let $x_1, x_2, .., x_n, ..$ be an enumeration of the rational points of [0,1]. This is possible since, as we have shown, the rationals are a countable set. We note that $\{f_n\}_{n\geq 1}$, evaluated at x_1, forms an infinite sequence of real numbers. Since A is closed and bounded, each $\{f_n\}$ must also be bounded, and so our sequence of real numbers, $\{f_n(x_1)\}$, is also bounded. By the Bolzano-Weierstrass Theorem, then, there exists a subsequence of our sequence of real numbers which converges. This is equivalent to stating that there is a subsequence of $\{f_n\}_{n\geq 1}$ which converges pointwise at x_1. For notational convenience, we label this sequence $f_{n_1(k)}$, where $n_1(k)$ is a strictly increasing function from the positive integers to the positive integers. With exactly the same argument, we can create a subsequence of that subsequence which converges at x_2, which we label $f_{n_2(k)}$. Since $f_{n_1(k)}$ converges at x_1 and $f_{n_2(k)}$ is a subsequence of $f_{n_1(k)}$, $f_{n_2(k)}$ must also converge at x_1.

We can continue this chain of subsequences, and so obtain a sequence, for each positive integer m, a subsequence $f_{n_m(k)}$ which converges at the rational points $x_1, x_2, .., x_m$, created in such a way that $f_{n_m(k)}$ is a subsequence of $f_{n_{(m-1)}(k)}$.
Thus, for any particular finite number of rational points in [0,1], we can find a subsequence which converges at those rational points. As pointed out earlier, this will not be enough to find a subsequence which converges on the entire interval. However, we have not yet used the hypothesis of equicontinuity.
Before doing that, we define a sequence $\{g_n\}_{n\geq 1}$, by making the n'th function in the sequence equal to the n'th function in the sequence $f_{n_n(k)}$. That is, the n'th function in g is equal to the n'th function of the n'th subsequence of the f_n's. We note that, for any given rational point x_i, g_n is a subsequence of $f_{n_i(k)}$ for all $n \geq i$, and so g_n converges at x_i. Thus, this sequence in fact converges at every single rational point on [0,1]. Since the elements of $\{g_n\}$ are all taken from subsequences of $\{f_n\}$, we note that it is also a subsequence of $\{f_n\}$.

At this point, it remains to show that g_n converges everywhere on [0,1], and also that it converges uniformly.

First, we will show that it is a Cauchy sequence. We consider an arbitrary $x \in$ [0,1]. We note immediately that, by the triangle inequality,:

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| + |g_m(x) - g_m(x_i)| \quad (3)$$

for any point x_i in [0,1]. Here, for the first time, we use equicontinuity. We can choose a δ such that $|x - x_i| < \delta$ implies both that $|g_n(x) - g_n(x_i)| < \frac{\epsilon}{3}$ and that $|g_m(x) - g_m(x_i)| < \frac{\epsilon}{3}$. This δ, we recall, is completely independent of m and n, and it is also completely independent of x, x_i. We note that the rational points are dense in the reals, and so we choose now x_i to be a rational point satisfying $|x - x_i| < \delta$. As for the middle term, g_n converges at x_i, and so g_n evaluated at x_i forms a Cauchy sequence, after we have already chosen x_i. Thus, $\exists N > 0$ such that $m, n > N$ forces the middle term to be less than $\frac{\epsilon}{3}$. And so, we have shown that $g_n(x)$ is itself a Cauchy sequence, that is, it converges pointwise everywhere on [0,1].

We need to show now that this convergence is uniform. That is, the convergence is essentially independent of x. The proof above of pointwise convergence depended on x.

Fortunately, it can actually be modified to prove not only pointwise convergence, but uniform convergence. Once again to start, we allow $\epsilon > 0$ to be given. Since A is equicontinuous, we can choose a δ independent of n and x such that $|x - x_i| < \delta$ implies that $|g_n(x) - g_n(x_i)| < \frac{\epsilon}{3}$ for all n. Now, we partition the interval into intervals of length $\frac{\delta}{2}$. We can now choose exactly one rational point in each such interval. We are now looking at a finite number of rational points, only. Since g converges at each rational point, for each rational point x_j which we are looking at, there exists an N_j such that $m, n > N_j$ implies that $|g_m(x_j) - g_n(x_j)| < \frac{\epsilon}{3}$. Now, we define N to be the maximum over all the N_j's, which exists, since we are taking the maximum over a finite set.
Having done this preparatory work, we will now show that in fact the convergence is uniform. For ϵ, δ, N and the set of rational points x_j with their associated partition as above, we continue. Note that for any $x \in$ [0,1], we can choose one of

our special rational points, x_j, that is within δ of x. Choosing this rational point, and forcing m, n to be strictly greater than N, we obtain:

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| + |g_m(x) - g_m(x_i)| \quad (4)$$

but $|g_n(x) - g_n(x_j)| < \frac{\epsilon}{3}$, $|g_m(x) - g_m(x_j)| < \frac{\epsilon}{3}$ since $|x - x_j| < \delta$. We also see that, $\forall x_j \in$ [0,1], $|g_n(x_j) - g_m(x_j)| < \frac{\epsilon}{3}$, since we have already imposed the restriction $m, n > N > N_j$. And so, $|g_n(x) - g_m(x)| < \epsilon$, as we wished to show. Since A is closed and this sequence converges, it must of course converge to a function in A.

Thus, from the assumptions that A is closed, bounded and equicontinuous, we have demonstrated for a general sequence the existence of a convergent subsequence with limit point in A. □

9 Closing Examples

Having proven Arzela-Ascoli's characterization of compact sets, it would be good to see if this characterization has any applications. In particular, are there any subsets of C[0,1] which we can prove are compact using this theorem, which might otherwise be difficult? To answer this, we first construct a sequence in C[0,1] which does not converge, and which is not equicontinuous.

Let $\{f_n\}_{n\geq 1}$ be a sequence, where f_n is a "smoothed out" triangle of height 1, base 2^{-n}, non-zero over $[1-2^{1-n}, 1-2^{-n}]$, and 0 everywhere else. The "smoothed out" condition means that we make these functions sufficiently nice to be differentiable everywhere, while still being very close approximations to triangles.

We can see immediately that this sequence cannot possibly converge, and in fact, the supports of the elements in the sequence are pairwise disjoint. We note, however, that the triangles become progressively sharper as the sequence continues - that is, the magnitudes of the derivatives increase, and clearly without bound. It is this unbounded increase in the derivative which allows an infinite sequence of pseudo-triangles to exist on a bounded domain, without their supports intersecting. Intuitively, then, one might expect that if we don't allow functions to have unbounded derivative (that is, we don't allow them to become very sharp), they might have to remain close to each other. This turns out to be the case.

Bounded Derivative Theorem. *Consider a sequence $\{f_n\}_{n\geq 1}$ in C[0,1] subjected to the restrictions that $||f_n|| \leq 1$, f_n is differentiable for all n, and finally there exists a positive constant C such that $||f_n'|| \leq C$ for all n. Then we may conclude that the completion of the set $\{f_n\}_{n\geq 1}$ is compact, and thus that the sequence has a convergent subsequence.*

Proof. To begin, we require a short lemma, which states that if a set A is equicontinuous, then so is its completion $\overline{A} = A \bigcup \partial A$. The proof is as follows:

Let $\epsilon > 0$ be given. Let $\delta > 0$ be a number such that $|x-y| < \delta \Rightarrow |g(x)-g(y)| < \frac{\epsilon}{3}$ $\forall\ g \in A$. We claim that this δ will have the property that, $\forall f \in \overline{A}$, $|x-y| < \delta$ $\Rightarrow \|f(x) - f(y)\| < \epsilon$. To prove this, consider an arbitrary $f \in \partial A$, then for any $g \in A$,

$$|f(x) - f(y)| \le |f(x) - g(x)| + |f(y) - g(y)| + |g(x) - g(y)| \tag{5}$$

For x, y satisfying $|x - y| < \delta$, the last term is less than $\frac{\epsilon}{3}$. Also, since f is the limit of at least one sequence in A, we may choose g to be arbitrarily close to f while still not being in the boundary. In particular, we can choose g in such a way that $\|f - g\| < \frac{\epsilon}{3}$, which implies that for all x, y in [0,1], the first two terms are individually less than $\frac{\epsilon}{3}$. This completes the proof.

We recall from elementary calculus the intermediate value theorem. It tells us that, $\forall x, y \in$ [0,1], there exists a $\phi \in [x,\ y]$ such that $f_n(x)$ - $f_n(y)$ = $(x$ - $y)f_n'(\phi)$. But $f_n'(\phi) \le C$, and so we note immediately that, $\forall x, y \in$ [0,1], $|f_n(x)$ - $f_n(y)| \le C|x - y|$. We also note that it is clear that our sequence is closed and bounded, by its definition. It remains to show only that it is also equicontinuous, which we are ready to do.
Let $\epsilon > 0$ be given. Define $\delta = \frac{\epsilon}{C}$. Then we have immediately that, if $|x - y| < \delta$, then $|f_n(x)$ - $f_n(y)| \le C|x - y| \le C\delta = \epsilon$. We note that this estimate is actually completely independent of n, and so we have shown that the sequence is in fact equicontinuous. By the lemma above, this implies that its completion is also equicontinuous. Thus, the set is compact, as we wished to show. □

This theorem allows us to prove a related theorem, as follows:

Bounded Norm Theorem. *Consider a sequence $\{f_n\}_{n\ge1}$ in C[0,1] and a positive constant K such that $\|f_n\| \le K\ \forall n$. Then the sequence $\{F_n\}_{n\ge1}$ defined by $F_n(x) = \int_0^x f_n(x)dx$ for $x \in$ [0,1] is in C[0,1] and has a uniformly convergent subsequence.*

Proof. Since each function f_n is a continuous function on [0,1], it is clear from elementary calculus that each function F_n must also be continuous functions on [0,1]. We note, by the fundamental theorem of calculus, that these functions are differentiable, and that the derivative of F_n is exactly f_n. We define a sequence of functions $\{g_n\}_{n\ge1}$ by $g_n = \frac{1}{K}F_n$. By assumption, $\|f_n\| \le K$, and so $\|g_n'\| \le 1$. Also, we note that $\|g_n\| \le 1$, since it is the integral of a function with a maximum of 1 over the interval [0,1]. And so, the sequence $\{g_n\}_{n\ge1}$ satisfies the conditions for the Bounded Derivative Theorem, above. Thus, it has a convergent subsequence, $\{g_{n(k)}\}_{k\ge1}$. But then, since the sequence of g_n's is simply a rescaling of the sequence of F_n's, we recall from elementary analysis that under the same index function n(k), the sequence $\{F_{n(k)}\}_{k\ge1}$ must also be a convergent sequence. Since that is clearly a subsequence of our original sequence by construction, we are done. □

References

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[3] Notes and Templates on LaTeX by Andrew Mauer-Oats.