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Chapter 9
Connected and Path Connected Metric Spaces

Consider the following subsets of $\mathbb{R}$:

$$S = [-1, 0] \cup [1, 2] \text{ and } T = [0, 1].$$

Notice that S is made up of two "parts" and that T consists of just one. This notion can be more precisely described using the following definition.

9.1 - Definition: A subset A of a metric space X is *not connected* (disconnected) if there are disjoint open sets U and V such that $A \subset U \cup V$, $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$.
If no such disjoint open sets U and V exist then A is *connected*.

9.2 - Examples:

1. Let $S = [-1, 0] \cup [1, 2] \subset \mathbb{R}$. Let $U = (-1.1, 0.1)$ and $V = (0.9, 2.2)$. U and V are disjoint, open subsets of $\mathbb{R}$. $S \subset U \cup V$, $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$. Thus, S is not connected.
2. Let $S = [-1, 0) \cup (0, 1] \subset \mathbb{R}$. Let $U = (-2, 0)$ and $V = (0, 2)$. U and V are disjoint, open subsets of $\mathbb{R}$. $S \subset U \cup V$, $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$. Therefore, S is not connected.

9.3 - Proposition: *The interval* $[a, b]$ *is connected.*

Before proceeding with the proof, recall the following:

The Intermediate Value Theorem: Suppose that f is a continuous function $f : [s, t] \to \mathbb{R}$. If N is a real number between $f(s)$ and $f(t)$, then $\exists c \in [s, t]$ such that $f(c) = N$.

We are now ready to prove the proposition.

Proof: Assume that $[a, b]$ is not connected. Then there exist some disjoint open sets, U and V, such that $[a, b] \subset U \cup V$, $[a, b] \cap U \neq \emptyset$ and $[a, b] \cap V \neq \emptyset$. Define a function,

$$f(x) = \begin{cases} 1 & \text{if } x \in [a, b] \cap U \\ -1 & \text{if } x \in [a, b] \cap V \end{cases}$$

Claim: *f is continuous.*
Let $\varepsilon > 0$ be given and let x_0 be in $[a, b] \subset U \cup V$. Because U is an open set, if x_0 is in U, then there is some $\delta > 0$ such that $B_\delta(x_0) \subset U$. For every x with $|x - x_0| < \delta$, we know that $x \in U$ and thus $f(x) = 1$. Therefore, for $|x - x_0| < \delta$, we have:

$$| f(x) - f(x_0)| = | 1 - 1| = 0 < \varepsilon.$$

Similarly, since V is an open set, if $x_0 \in V$, then there is some $\delta > 0$ such that $B_\delta(x_0) \subset V$. So, whenever $| x - x_0| < \delta$, we have:

$$| f(x) - f(x_0)| = | - 1 - (-1)| = 0 < \varepsilon.$$

Thus, f is a continuous function from $[a, b]$ to $\mathbb{R}$.

Now, the range of f is $\{-1, 1\}$ which, because f is continuous, is a contradiction to the intermediate value theorem. Therefore, the assumption that $[a, b]$ is not connected is wrong. Hence, $[a, b]$ is connected.

9.4 - Theorem: *Let (X, ρ) and (Y, σ) be metric spaces. Suppose that $f : (X, \rho) \to (Y, \sigma)$ is a continuous function. If X is connected, then the image, $f(X)$ is connected. (In other words, the continuous image of a connected set is connected.)*

Proof: Assume that $f(X)$ is not connected. Then there exist some disjoint open sets, U and V, such that $f(X) \subset U \cup V$, $f(X) \cap U \neq \emptyset$ and $f(X) \cap V \neq \emptyset$. Let $S = f^{-1}(U)$ and $T = f^{-1}(V)$.

1. Since f is continuous, the inverse image of an open set is open. Therefore, S and T are open.

2. $S \cap T = f^{-1}(U) \cap f^{-1}(V) = \emptyset$.
   To show this, assume that $x \in f^{-1}(U) \cap f^{-1}(V)$. Then, $x \in f^{-1}(U)$ and $x \in f^{-1}(V)$. This implies that $f(x) \in U$ and $f(x) \in V$ which means that $f(x) \in U \cap V$. Thus, $U \cap V \neq \emptyset$ which is a contradiction. So, $S \cap T = \emptyset$.

3. $X \subset S \cup T$.
   Assume that there exists an $x \in X$ with $x \notin S \cup T$. So, $x \notin f^{-1}(U) \cup f^{-1}(V)$. Then, as $x \notin f^{-1}(U)$ and $x \notin f^{-1}(V)$ we get that $f(x) \notin U$ and $f(x) \notin V$. Therefore, $f(x) \notin U \cup V$. As this is a contradiction, $X \subset S \cup T$.

4. $X \cap S \neq \emptyset$ and $X \cap T \neq \emptyset$.
   First we show that $X \cap S \neq \emptyset$. To do this, suppose $X \cap S = X \cap f^{-1}(U) = \emptyset$. Therefore, if $x \in X$ then $x \notin f^{-1}(U)$. Hence, if $x \in X$, then $f(x) \notin U$. Thus, $f(X) \cap U = \emptyset$. This is a contradiction. Therefore, $X \cap S \neq \emptyset$. By the same argument, $X \cap T \neq \emptyset$.

As a summary of the four points above, X is NOT connected. This contradicts the condition that X is connected. This then implies that the original assumption that $f(X)$ is not connected must be incorrect. Therefore, $f(X)$ is connected.

9.5 - Example: *A Path*

A path in a metric space is a continous image of the interval $[0, 1]$. Hence, by propositions 9.3 and 9.4, a path is always connected. In particular, if $f : [0, 1] \to \mathbb{R}^n$ is continuous, then the graph $G(f) = \{(x, f(x)) \ : \ 0 \leq x \leq 1\}$ is connected.

9.6 - Definition: A subset S of a metric space is *path connected* if for all $x, y \in S$ there is a path in S connecting x and y.

9.7 - Proposition: *Every path connected set is connected.*

Proof: Let S be path connected. Assume that S is not connected. Therefore, there exist open sets U and V such that $U \cap V = \emptyset$, $S \subset U \cup V$, $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$. Pick $u \in S \cap U$, $v \in S \cap V$. Since S is path connected, there is a path P from u to v in S. $P \subset U \cup V$. Also, $u \in P \cap U \Rightarrow P \cap U \neq \emptyset$ and $v \in P \cap V \Rightarrow P \cap V \neq \emptyset$.

Thus, since U and V are open, disjoint sets such that $P \subset U \cup V$, $P \cap U \neq \emptyset$ and $P \cap V \neq \emptyset$, P is not connected. However, by example 9.5, we know that every path is connected. So, we have reached a contradiction. Thus, the assumption was incorrect and so S is connected.

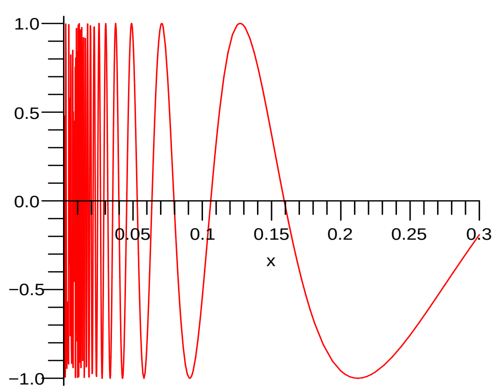
9.8 - Exercise: *Show that the only connected subsets of* $\mathbb{R}$ *are intervals.*

Solution: Consider an interval $I \subset \mathbb{R}$. Let x, $y \in I$ be given. Then the path $f : [0, 1] \to I$ with $f(t) = (y - x)t + x$ will connect x to y. Therefore, I is path connected and thus connected (by Prop. 9.7).

To complete the proof, we now show that any subset of $\mathbb{R}$ that is not an interval is not connected. We proceed by contradiction. Let $J \subset \mathbb{R}$ be connected. Suppose that J is not an interval in $\mathbb{R}$. Since it is not an interval, there are points x, y, $a \in \mathbb{R}$ such that x, $y \in J$, $a \notin J$ and $x < a < y$. Let $U = (-\infty, a)$ and let $V = (a, \infty)$. U and V are open, disjoint subsets of $\mathbb{R}$. $J \subset U \cup V$. As well, since $x \in U$, $J \cap U \neq \emptyset$ and since $y \in V$, $J \cap V \neq \emptyset$. Thus J is not connected. This is a contradiction and so the assumption that J is not an interval is incorrect.

9.9 - Example: *There exist sets which are connected but not path connected.*

To see this, we consider the set $S = \{(0, y) : |y| \leq 1\} \cup \{(x, \sin\frac{1}{x}) : 0 < x \leq 1\}$. Part of this set is pictured below.



S is connected. To see this, note that both $\{(0, y) : |y| \leq 1\}$ and $\{(x, \sin\frac{1}{x}) : 0 < x \leq 1\}$ are path connected and thus connected. Now assume that S is not connected and thus that there exist two open, disjoint sets U and V which cover S and such that $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$. Then we must have that $\{(0, y) : |y| \leq 1\}$ is completely contained in one of the sets, say U, and that $\{(x, \sin\frac{1}{x}) : 0 < x \leq 1\}$ is completely contained in V. (If this were not the case then we would contradict the connectedness of $\{(0, y) : |y| \leq 1\}$ or the connectedness of $\{(x, \sin\frac{1}{x}) : 0 < x \leq 1\}$). However, any open set that contains points in $\{(0, y) : |y| \leq 1\}$ will also contain points in $\{(x, \sin\frac{1}{x}) : 0 < x \leq 1\}$. This then contradicts that U and V are disjoint. Thus, there cannot be two disjoint, open sets that cover S with $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$. As a result, S must be connected.

S is not path connected. To explain why, we proceed by contradiction. Suppose that S is path connected. Then there exists a continuous function $f : [0, 1] \to S$ with $f(0) = (0, 0)$ and $f(1) = (1, \sin 1)$. Now let $Y = \{(0, y) : |y| \leq 1\}$ and let $a = \sup\{x : f(x) \in Y\}$. Since $a = \sup\{x : f(x) \in Y\}$, we have that for any $\varepsilon > 0$, $a + \varepsilon \notin Y$.

Since f is a continuous function, there is a $\delta > 0$ such that $||f(x) - f(a)|| \leq \frac{1}{2}$ whenever $|x - a| \leq \delta$. Let $b = a + \delta$. Since $a + \delta \notin Y$, $f(b) = (u, \sin\frac{1}{u})$ for some $u \in [0, 1]$. Also, any c with $a < c < b$ has $f(c) = (t, \sin\frac{1}{t})$ for some $t \in [0, 1]$.

Because f is continuous, $f([c, b])$ is connected. However, removing any point from $f([c, b])$ will disconnect it. Therefore, the set $A = \{(x, \sin\frac{1}{x}) : t \leq x \leq u\}$ is contained in $f([c, b])$. So, if c is chosen close enough to a so that the graph of $\sin\frac{1}{x}$ completes a full oscillation in the interval $[t, u]$, then there are $x_1, x_2 \in [t, u]$ with $\sin\frac{1}{x_1} = 1$ and $\sin\frac{1}{x_2} = -1$. Hence, there must be a point $\mathbf{v} = (v, \sin\frac{1}{v})$ where v is between x_1 and x_2 and where $||f(a) - \mathbf{v}|| > 1$. As $\mathbf{v} \in f([c, b])$, there must be some $d \in [c, b]$ such that $f(d) = \mathbf{v}$. This means that $|d - a| \leq \delta$ but $||f(a) - \mathbf{v}|| > \frac{1}{2}$ which contradicts the continuity of f. Thus, the original assumption was wrong and S is not path connected.