Chapter 9
Connected and Path Connected Metric Spaces

Consider the following subsets of $\mathbb{R}$:

$$S = [-1, 0] \cup [1, 2] \text{ and } T = [0, 1].$$

Notice that $S$ is made up of two “parts” and that $T$ consists of just one. This notion can be more precisely described using the following definition.

9.1 - Definition: A subset $A$ of a metric space $X$ is not connected (disconnected) if there are disjoint open sets $U$ and $V$ such that $A \subseteq U \cup V$, $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$. If no such disjoint open sets $U$ and $V$ exist then $A$ is connected.

9.2 - Examples:

1. Let $S = [-1, 0] \cup [1, 2] \subseteq \mathbb{R}$. Let $U = (-1.1, 0.1)$ and $V = (0.9, 2.2)$. $U$ and $V$ are disjoint, open subsets of $\mathbb{R}$. $S \subseteq U \cup V$, $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$. Thus, $S$ is not connected.

2. Let $S = [-1, 0] \cup (0, 1] \subseteq \mathbb{R}$. Let $U = (-2, 0)$ and $V = (0, 2)$. $U$ and $V$ are disjoint, open subsets of $\mathbb{R}$. $S \subseteq U \cup V$, $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$. Therefore, $S$ is not connected.

9.3 - Proposition: The interval $[a, b]$ is connected.

Before proceeding with the proof, recall the following:

The Intermediate Value Theorem: Suppose that $f$ is a continuous function $f : [s, t] \to \mathbb{R}$. If $N$ is a real number between $f(s)$ and $f(t)$, then $\exists c \in [s, t]$ such that $f(c) = N$.

We are now ready to prove the proposition.

Proof: Assume that $[a, b]$ is not connected. Then there exist some disjoint open sets, $U$ and $V$, such that $[a, b] \subseteq U \cup V$, $[a, b] \cap U \neq \emptyset$ and $[a, b] \cap V \neq \emptyset$. Define a function,

$$f(x) = \begin{cases} 
1 & \text{if } x \in [a, b] \cap U \\
-1 & \text{if } x \in [a, b] \cap V
\end{cases}$$

Claim: $f$ is continuous.

Let $\varepsilon > 0$ be given and let $x_0$ be in $[a, b] \cap U$. Because $U$ is an open set, if $x_0$ is in $U$, then there is some $\delta > 0$ such that $B_\delta(x_0) \subseteq U$. For every $x$ with $|x - x_0| < \delta$, we know that $x \in U$ and thus $f(x) = 1$. Therefore, for $|x - x_0| < \delta$, we have:
Similarly, since $V$ is an open set, if $x_0 \in V$, then there is some $\delta > 0$ such that $B_\delta(x_0) \subset V$. So, whenever $|x - x_0| < \delta$, we have:

$$| f(x) - f(x_0)| = |1 - 1| = 0 < \varepsilon.$$ 

Thus, $f$ is a continuous function from $[a, b]$ to $\mathbb{R}$.

Now, the range of $f$ is $\{-1, 1\}$ which, because $f$ is continuous, is a contradiction to the intermediate value theorem. Therefore, the assumption that $[a, b]$ is not connected is wrong. Hence, $[a, b]$ is connected.

**9.4 - Theorem:** Let $(X, \rho)$ and $(Y, \sigma)$ be metric spaces. Suppose that $f : (X, \rho) \rightarrow (Y, \sigma)$ is a continuous function. If $X$ is connected, then the image, $f(X)$ is connected. (In other words, the continuous image of a connected set is connected.)

**Proof:** Assume that $f(X)$ is not connected. Then there exist some disjoint open sets, $U$ and $V$, such that $f(X) \subset U \cup V$, $f(X) \cap U \neq \emptyset$ and $f(X) \cap V \neq \emptyset$. Let $S = f^{-1}(U)$ and $T = f^{-1}(V)$.

1. Since $f$ is continuous, the inverse image of an open set is open. Therefore, $S$ and $T$ are open.

2. $S \cap T = f^{-1}(U) \cap f^{-1}(V) = \emptyset$.

To show this, assume that $x \in f^{-1}(U) \cap f^{-1}(V)$. Then, $x \in f^{-1}(U)$ and $x \in f^{-1}(V)$. This implies that $f(x) \in U$ and $f(x) \in V$ which means that $f(x) \in U \cap V$. Thus, $U \cap V \neq \emptyset$ which is a contradiction. So, $S \cap T = \emptyset$.

3. $X \subset S \cup T$.

Assume that there exists an $x \in X$ with $x \notin S \cup T$. So, $x \notin f^{-1}(U) \cup f^{-1}(V)$. Then, as $x \notin f^{-1}(U)$ and $x \notin f^{-1}(V)$ we get that $f(x) \notin U$ and $f(x) \notin V$. Therefore, $f(x) \notin U \cup V$. As this is a contradiction, $X \subset S \cup T$.

4. $X \cap S \neq \emptyset$ and $X \cap T \neq \emptyset$.

First we show that $X \cap S \neq \emptyset$. To do this, suppose $X \cap S = X \cap f^{-1}(U) = \emptyset$. Therefore, if $x \in X$ then $x \notin f^{-1}(U)$. Hence, if $x \in X$, then $f(x) \notin U$. Thus, $f(X) \cap U = \emptyset$. This is a contradiction. Therefore, $X \cap S \neq \emptyset$. By the same argument, $X \cap T \neq \emptyset$.

As a summary of the four points above, $X$ is NOT connected. This contradicts the condition that $X$ is connected. This then implies that the original assumption that $f(X)$ is not connected must be incorrect. Therefore, $f(X)$ is connected.

**9.5 - Example:** A *Path*

A path in a metric space is a continuous image of the interval $[0, 1]$. Hence, by propositions 9.3 and 9.4, a path is always connected. In particular, if $f : [0, 1] \rightarrow \mathbb{R}^n$ is continuous, then
the graph $G(f) = \{(x, f(x)) : 0 \leq x \leq 1\}$ is connected.

**9.6 - Definition:** A subset $S$ of a metric space is path connected if for all $x, y \in S$ there is a path in $S$ connecting $x$ and $y$.

**9.7 - Proposition:** Every path connected set is connected.

**Proof:** Let $S$ be path connected. Assume that $S$ is not connected. Therefore, there exist open sets $U$ and $V$ such that $U \cap V = \emptyset$, $S \subset U \cup V$, $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$. Pick $u \in S \cap U$, $v \in S \cap V$. Since $S$ is path connected, there is a path $P$ from $u$ to $v$ in $S$. $P \subset U \cup V$. Also, $u \in P \cap U \Rightarrow P \cap U \neq \emptyset$ and $v \in P \cap V \Rightarrow P \cap V \neq \emptyset$.

Thus, since $U$ and $V$ are open, disjoint sets such that $P \subset U \cup V$, $P \cap U \neq \emptyset$ and $P \cap V \neq \emptyset$, $P$ is not connected. However, by example 9.5, we know that every path is connected. So, we have reached a contradiction. Thus, the assumption was incorrect and so $S$ is connected.

**9.8 - Exercise:** Show that the only connected subsets of $\mathbb{R}$ are intervals.

**Solution:** Consider an interval $I \subset \mathbb{R}$. Let $x, y \in I$ be given. Then the path $f : [0, 1] \rightarrow I$ with $f(t) = (y - x)t + x$ will connect $x$ to $y$. Therefore, $I$ is path connected and thus connected (by Prop. 9.7).

To complete the proof, we now show that any subset of $\mathbb{R}$ that is not an interval is not connected. We proceed by contradiction. Let $J \subset \mathbb{R}$ be connected. Suppose that $J$ is not an interval in $\mathbb{R}$. Since it is not an interval, there are points $x, y, a \in \mathbb{R}$ such that $x, y \in J$, $a \notin J$ and $x < a < y$. Let $U = (-\infty, a)$ and let $V = (a, \infty)$. $U$ and $V$ are open, disjoint subsets of $\mathbb{R}$. $J \subset U \cup V$. As well, since $x \in U$, $J \cap U \neq \emptyset$ and since $y \in V$, $J \cap V \neq \emptyset$. Thus $J$ is not connected. This is a contradiction and so the assumption that $J$ is not an interval is incorrect.

**9.9 - Example:** There exist sets which are connected but not path connected.

To see this, we consider the set $S = \{(0, y) : |y| \leq 1\} \cup \{(x, \sin \frac{1}{x}) : 0 < x \leq 1\}$. Part of this set is pictured below.
S is connected. To see this, note that both \( \{(0, y) : |y| \leq 1\} \) and \( \{(x, \sin \frac{1}{x}) : 0 < x \leq 1\} \) are path connected and thus connected. Now assume that \( S \) is not connected and thus that there exist two open, disjoint sets \( U \) and \( V \) which cover \( S \) and such that \( S \cap U \neq \emptyset \) and \( S \cap V \neq \emptyset \). Then we must have that \( \{(0, y) : |y| \leq 1\} \) is completely contained in one of the sets, say \( U \), and that \( \{(x, \sin \frac{1}{x}) : 0 < x \leq 1\} \) is completely contained in \( V \). (If this were not the case then we would contradict the connectedness of \( \{(0, y) : |y| \leq 1\} \) or the connectedness of \( \{(x, \sin \frac{1}{x}) : 0 < x \leq 1\} \). However, any open set that contains points in \( \{(0, y) : |y| \leq 1\} \) will also contain points in \( \{(x, \sin \frac{1}{x}) : 0 < x \leq 1\} \). This then contradicts that \( U \) and \( V \) are disjoint. Thus, there cannot be two disjoint, open sets that cover \( S \) with \( S \cap U \neq \emptyset \) and \( S \cap V \neq \emptyset \). As a result, \( S \) must be connected.

\( S \) is not path connected. To explain why, we proceed by contradiction. Suppose that \( S \) is path connected. Then there exists a continuous function \( f : [0, 1] \to S \) with \( f(0) = (0, 0) \) and \( f(1) = (1, \sin 1) \). Now let \( Y = \{(0, y) : |y| \leq 1\} \) and let \( a = \sup \{x : f(x) \in Y\} \). Since \( a = \sup \{x : f(x) \in Y\} \), we have that for any \( \varepsilon > 0 \), \( a + \varepsilon \notin Y \).

Since \( f \) is a continuous function, there is a \( \delta > 0 \) such that \( ||f(x) - f(a)|| \leq \frac{1}{2} \) whenever \( |x - a| \leq \delta \). Let \( b = a + \delta \). Since \( a + \delta \notin Y \), \( f(b) = (u, \sin \frac{1}{u}) \) for some \( u \in [0, 1] \). Also, any \( c \) with \( a < c < b \) has \( f(c) = (t, \sin \frac{1}{t}) \) for some \( t \in [0, 1] \).

Because \( f \) is continuous, \( f([c, b]) \) is connected. However, removing any point from \( f([c, b]) \) will disconnect it. Therefore, the set \( A = \{(x, \sin \frac{1}{x}) : t \leq x \leq u\} \) is contained in \( f([c, b]) \). So, if \( c \) is chosen close enough to \( a \) so that the graph of \( \sin \frac{1}{x} \) completes a full oscillation in the interval \([t, u]\), then there are \( x_1, x_2 \in [t, u] \) with \( \sin \frac{1}{x_1} = 1 \) and \( \sin \frac{1}{x_2} = -1 \). Hence, there must be a point \( \mathbf{v} = (v, \sin \frac{1}{v}) \) where \( v \) is between \( x_1 \) and \( x_2 \) and where \( ||f(a) - \mathbf{v}|| > 1 \). As \( \mathbf{v} \in f([c, b]) \), there must be some \( d \in [c, b] \) such that \( f(d) = \mathbf{v} \). This means that \( |d - a| \leq \delta \) but \( ||f(a) - \mathbf{v}|| > \frac{1}{2} \) which contradicts the continuity of \( f \). Thus, the original assumption was wrong and \( S \) is not path connected.