1. Suppose we have a triangle with vertices labelled 1, 2 and 3.

A particle initially starts at a randomly chosen vertex, and then makes 2 moves. On each move the particle chooses an adjacent vertex at random then moves to the chosen vertex.

Let $X_i$ = # of times particle visits vertex $i$ (including the initial starting vertex), $i = 1, 2, 3$

(a) Find the joint pmf of $(X_1, X_2, X_3)^T$. Make sure to specify the support of the distribution.

Sol. Make table. (The probability of each outcome is $\frac{1}{12}$)

<table>
<thead>
<tr>
<th>Outcome</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1 2 3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1 3 1</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1 3 2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2 1 2</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2 1 3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2 3 2</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2 3 1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3 2 3</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3 2 1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3 1 3</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3 1 2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Let $p(x_1, x_2, x_3)$ denote the joint pmf.

- $p(1, 1, 1) = \frac{1}{12}$
- $p(2, 1, 0) = \frac{1}{12}$
- $p(2, 0, 1) = \frac{1}{12}$
- $p(1, 2, 0) = \frac{1}{12}$
- $p(0, 2, 1) = \frac{1}{12}$
- $p(1, 0, 2) = \frac{1}{12}$
- $p(x_1, x_2, x_3) = 0$ otherwise.

(b) Are $X_1$ and $X_2$ independent? Prove your answer.

Sol. $X_1$ and $X_2$ are not independent. For example,

$p(X_1 = 1, X_2 = 2) = \frac{1}{12}$

$p(X_1 = 1) = \frac{3}{6}$, $p(X_2 = 2) = \frac{1}{6}$ and $p(X_1 = 1)p(X_2 = 2) = \frac{2}{6} \cdot \frac{1}{6} = \frac{1}{12}$

:. $p(X_1 = 1, X_2 = 2) \neq p(X_1 = 1)p(X_2 = 2)$. 
Suppose $X_1$ and $X_2$ are jointly continuous with joint pdf
\[
f(x_1, x_2) = \begin{cases} 
\frac{3x_1^3}{x_2^4} e^{-x_1} & 0 < x_1 < x_2 < \infty \\
0 & \text{otherwise}
\end{cases}
\]

Find $\rho(X_1, X_2)$, the correlation coefficient between $X_1$ and $X_2$.

**Hint:** Integrate with respect to $x_2$ first.

**Sol:** First, note that the marginal pdf of $X_1$ is
\[
f_{x_1}(x_1) = \int_0^\infty \frac{3x_1^3}{x_2^4} e^{-x_1} dx_2 = 3x_1^3 e^{-x_1} \int_0^{\infty} x_2^{-4} dx_2
\]
\[= 3x_1^3 e^{-x_1} \left( \frac{x_2^{-3}}{-3} \right)^{\infty}_{x_1}
\]

1.e., $X_1 \sim \text{Exponential}(1)$, so $E[X_1] = \frac{1}{\lambda}, \ Var(X_1) = \frac{1}{\lambda^2}$

Next, we have
\[
E[X_1^n] = \int_0^\infty \int_0^\infty x_1^n \frac{3x_1^3}{x_2^4} e^{-x_1} dx_2 dx_1 = 3x_1^3 e^{-x_1} \int_0^{\infty} x_2^{-4} dx_2 \int_0^{\infty} x_1^n dx_1
\]
\[= 3x_1^3 e^{-x_1} \int_0^{\infty} x_1^n \left( \frac{x_2^{-3}}{-3} \right)^{\infty}_{x_1}
\]
\[= \frac{3}{3-n} \int_0^{\infty} x_1^n e^{-x_1} dx_1 = \frac{1}{\Gamma(n+1)} \left( \frac{x_1^{n+1} e^{-x_1}}{\Gamma(n+1)} \right) \text{ for } n = 1 \text{ or } 2
\]
\[= \frac{3 \Gamma(n+1)}{3-n} \int_0^{\infty} \left( \frac{1}{\Gamma(n+1)} x_1^{n+1} e^{-x_1} \right) dx_1
\]
\[= \frac{3 n!}{3-n}
\]
\[= \begin{cases} \frac{3}{2} & \text{if } n = 1 \\
6 & \text{if } n = 2
\end{cases}
\]

So $E[X_2] = \frac{3}{2}, E[X_2^2] = 6, \ Var(X_2) = 6 - \frac{9}{4} = \frac{15}{4}$
\[
E(X_1X_2) = \int_{0}^{\infty} \int_{0}^{\infty} x_1 x_2 \frac{3 x_1^3}{x_2^4} e^{-x_1} d_2 dx_1 \\
= \int_{0}^{\infty} 3 x_1^4 e^{-x_1} \int_{0}^{\infty} x_2^{-3} dx_2 dx_1 \\
= \frac{3}{2} \int_{0}^{\infty} x_1^2 e^{-x_1} dx_1 \\
= \frac{3 T(3)}{2} \int_{0}^{\infty} \frac{x_1^{3-1} e^{-x_1} dx_1}{\Gamma(3, 1)} \text{ density} \\
= 3
\]

Then \(\text{Cov}(X_1, X_2) = 3 - (1)(\frac{3}{2}) = \frac{3}{2}\) and

\[
\rho(X_1, X_2) = \frac{3/2}{\sqrt{(1)(15/4)}} = \frac{3/2}{\sqrt{15}/2} = \frac{3}{\sqrt{15}} = \sqrt{\frac{3}{5}}.
\]
Consider a square with vertices labelled 1, 2, 3, 4 (moving in a clockwise direction). A particle starts at vertex 1. On each move it chooses an adjacent vertex at random and moves to the chosen vertex. It repeatedly makes moves.

(a) Find the expected number of moves to reach vertex 2 for the first time, the expected number of moves to reach vertex 3 for the first time, and the expected number of moves to return to vertex 1. Hint: Define $M_{ij}$ to be the expected number of moves to reach vertex $j$ for the first time, starting from vertex $i$ (if $i = j$, $M_{ii}$ is the expected number of moves to return to vertex $i$ starting at vertex $i$).

S01. Condition on the first move of the particle. We have

\[ M_{12} = (1) \frac{1}{2} + (1 + M_{42}) \frac{1}{2} \]
\[ M_{13} = (1 + M_{23}) \frac{1}{2} + (1 + M_{43}) \frac{1}{2} \]

By symmetry, $M_{42} = M_{13}$, $M_{23} = M_{12}$, $M_{43} = M_{12}$. Then

(i) \[ M_{12} = \frac{1}{2} + (1 + M_{13}) \frac{1}{2} = 1 + \frac{1}{2} M_{13} \]
(ii) \[ M_{13} = (1 + M_{12}) \frac{1}{2} + (1 + M_{12}) \frac{1}{2} = 1 + M_{12} \]

Plugging (ii) into (i) gives

\[ M_{12} = 1 + \frac{1}{2} (1 + M_{12}) \]

or \[ \frac{1}{2} M_{12} = \frac{3}{2} \]

or \[ M_{12} = 3 \]

Then \[ M_{13} = 1 + M_{12} = 4 \]

For $M_{11}$, again conditioning on the first move, we have

\[ M_{11} = (1 + M_{21}) \frac{1}{2} + (1 + M_{41}) \frac{1}{2} \]

By symmetry, $M_{21} = M_{12}$ and $M_{41} = M_{12}$. So, we get

\[ M_{11} = (1 + M_{12}) \frac{1}{2} + (1 + M_{12}) \frac{1}{2} = 1 + M_{12} = 1 + 3 = 4 \]

What is the expected number of moves until vertices 2 and 3 have both been reached?

Hint: Define \( M_{ijk} \), \( i, j, k \) distinct, to be the expected number of moves until both vertices \( j \) and \( k \) have been reached starting from vertex \( i \).

So, we want \( M_{1,23} \). Conditioning on first move, we have

\[
M_{1,23} = \left(1 + M_{23}\right)^{\frac{1}{2}} + \left(1 + M_{4,23}\right)^{\frac{1}{2}}
\]

By symmetry, \( M_{4,23} = M_{1,23} \) and \( M_{23} = M_{12} = 3 \), so

\[
M_{1,23} = \left(1 + 3\right)^{\frac{1}{2}} + \left(1 + M_{1,23}\right)^{\frac{1}{2}}, \text{ or }
\]

\[
\frac{1}{2} M_{1,23} = 2 + \frac{1}{2}, \text{ or } M_{1,23} = 4 + 1 = 5.
\]