Final Exam Practice Problems (Part 1)

In this part 1, there are 3 practice problems focused on the material in Chapter II, though material from other chapters are potentially incorporated as well.

1. Let \( X \) be a continuous random variable with pdf
   \[
   f(x) = \begin{cases} 
   2(1-x) & 0 < x < 1 \\
   0 & \text{otherwise}
   \end{cases}
   \]

   (a) Show that the mgf of \( X \), \( M_X(t) \), can be written as
   \[
   M_X(t) = 1 + \sum_{n=1}^{\infty} \left( \frac{t}{n+1} \right) \frac{t^n}{n!} .
   \]

   **Solution:**
   Write \( M_X(t) \) as
   \[
   M_X(t) = E[e^{tx}] = E\left[ \sum_{n=0}^{\infty} \left( \frac{tx}{n!} \right)^n \right] = \sum_{n=0}^{\infty} \frac{E[X^n]}{n!} t^n
   \]
   \[
   E(X^n) = \int_0^1 x^n (1-x) \, dx
   \]
   \[
   = 2 \frac{n!}{(n+2)!} \frac{\Gamma(n+1)\Gamma(2)}{\Gamma(n+1+2)} \frac{\Gamma(n+1+2)}{\Gamma(n+1)\Gamma(2)} \frac{\Gamma(n+1+1)}{\Gamma(n+1)\Gamma(2)} \frac{(1-x)^{n+1}}{n!} \, dx
   \]
   \[
   = 2 \frac{n!}{(n+2)!} .
   \]
   Then
   \[
   M_X(t) = \sum_{n=0}^{\infty} \frac{2 n!}{(n+2)!} t^n = 1 + \sum_{n=1}^{\infty} \frac{2}{(n+2)!} t^n
   \]
   Now note that \( \frac{n!}{\prod_{k=0}^{n-1} \left( \frac{k+1}{k+3} \right)} = \frac{2}{(n+2)!} \) and \( \frac{n!}{\prod_{k=0}^{n-1} \left( \frac{k+1}{k+3} \right)} \frac{1}{n!} = \frac{2}{(n+2)!} .
   
   (b) Let \( Y = \ln X \). Find \( M_Y(t) \), the mgf of \( Y \), for \( t > -1 \).

   **Solution:**
   \[
   M_Y(t) = E[e^{ty}] = E[e^{t \ln x}] = E[e^{inx}] = E[X^t] ,
   \]
   So
   \[
   M_Y(t) = E[X^t] = \int_0^1 x^t (1-x) \, dx = 2 \frac{\Gamma(t+1)\Gamma(3)}{\Gamma(t+3)} \frac{\Gamma(t+1+2)}{\Gamma(t+1)\Gamma(2)} \frac{x^t (1-x)}{n!} \, dx
   \]
   \[
   = \frac{2 \Gamma(t+1)}{\Gamma(t+3)} = \frac{2}{(t+2)(t+1)} .
   \]
Let $X_1, X_2, \ldots$ are independent Uniform $(0, 1)$ random variables.

Let $M_n$ to be the $n$th order statistic of $X_1, \ldots, X_{2n-1}$ (note that $M_n$ is the sample median of $X_1, \ldots, X_{2n-1}$).

(a) Show that $M_n \rightarrow \frac{1}{2}$ in mean square. *Hint:* $M_n$ has a beta distribution.

So $M_n \rightarrow \frac{1}{2}$ in mean square.

The pdf of $M_n$ is
\[
\frac{(2n-1)!}{(n-1)!n!} x^{n-1} (1-x)^{n-1} \quad 0 < x < 1,
\]
which is a Beta$(n,n)$ density.

So $E[M_n] = \frac{n}{n+n} = \frac{1}{2}, \quad Var(M_n) = \frac{nn}{(n+n)^2(n+n+1)} = \frac{1}{4(2n+1)}$

Now,
\[
E[|M_n - \frac{1}{2}|^2] = Var(M_n) = \frac{1}{4(2n+1)} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\]

\[ \therefore \quad M_n \rightarrow \frac{1}{2} \quad \text{in mean square}. \]

(b) Show that $M_n^2 \rightarrow \frac{1}{2}$ almost surely.

So $M_n \rightarrow \frac{1}{2}$ almost surely.

Let $\varepsilon > 0$ be given.

\[
P( |M_n^2 - \frac{1}{2}| > \varepsilon ) \leq \frac{Var(M_n^4)}{\varepsilon^2}
\]

by Chebyshev's inequality,

\[
= \frac{1}{4(2n+1)^2 \varepsilon^2}
\]

Thus, $\sum_{n=0}^{\infty} P( |M_n^2 - \frac{1}{2}| > \varepsilon ) < \infty$.

Then by the sufficient condition from the lecture, $M_n^2 \rightarrow \frac{1}{2}$ almost surely.
Suppose $X_1, X_2, \ldots$ are independent $N(0, 1)$ random variables.

Let $Y_i = e^{X_i}, \ i \geq 1$ (the $Y_i$'s are called log-normal random variables).

(a) Find the mean and variance of $Y_i$.

**Solution**. Note that $E[Y_i] = M_X(1)$ and $E[Y_i^2] = M_X(2)$, where $M_X(t)$ is the mgf of $X_i$. Since $X_i \sim N(0, 1)$, $M_X(t) = e^{t^2/2}$.

So $E[Y_i] = e^{1/2} = e^{1/2} = \sqrt{e}$

$E[Y_i^2] = e^{2/2} = e^{1/2} = e^2$

Then $Var(Y_i) = e^2 - e = e(e-1)$

(b) Show that $\frac{1}{n} \sum_{i=1}^{n} e^{X_i}$ and $e^{\frac{1}{n} \sum_{i=1}^{n} X_i}$ both converge to constants almost surely. What are these limiting constants?

**Solution**. Let $\bar{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i$ and $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

Then $\frac{1}{n} \sum_{i=1}^{n} e^{X_i} = \bar{Y}_n$. Since $Y_1, Y_2, \ldots$ are i.i.d. with finite mean $\sqrt{e}$ and finite variance $e(e-1)$, by the strong law of large numbers,

$\bar{Y}_n \to E[Y_i] = \sqrt{e}$ almost surely.

Next, $e^{\frac{1}{n} \sum_{i=1}^{n} X_i} = e^{\bar{X}_n}$, By the SLLN again, $\bar{X}_n \to E[X_i] = 0$ almost surely. Since $e^x$ is a continuous function, we have that $e^{\bar{X}_n} \to e^0 = 1$ almost surely.

(c) Use the central limit theorem to approximate

$p\left(\frac{1}{n} \sum_{i=1}^{n} e^{X_i} > \sqrt{e} + \frac{1}{\sqrt{n}}\right)$, for $n$ large

**Solution**. $p\left(\frac{1}{n} \sum_{i=1}^{n} e^{X_i} > \sqrt{e} + \frac{1}{\sqrt{n}}\right) = p\left(\bar{Y}_n > \sqrt{e} + \frac{1}{\sqrt{n}}\right)$, and note that $E[\bar{Y}_n] = \sqrt{e}$ and $Var(\bar{Y}_n) = e(e-1)$. Then

$p\left(\bar{Y}_n > \sqrt{e} + \frac{1}{\sqrt{n}}\right) = p\left(\frac{\bar{Y}_n - \sqrt{e}}{\sqrt{e(e-1)/n}} > \frac{1}{\sqrt{n \sqrt{e(e-1)/n}}}\right)$

$\sim N(0, 1)$

$\approx p\left(Z > \frac{1}{\sqrt{e(e-1)}}\right)$, where $Z \sim N(0, 1)$

$= 1 - \Phi\left(\frac{1}{\sqrt{e(e-1)}}\right) = 1 - \Phi(0.463) = 0.322$
Suppose we have a triangle with vertices labelled 1, 2 and 3.

A particle initially starts at a randomly chosen vertex, and then makes 2 moves. On each move the particle chooses an adjacent vertex at random then moves to the chosen vertex.

Let \( X_i = \# \) of times particle visits vertex \( i \) (including the initial starting vertex).

### i = 1, 2, 3

(a) Find the joint pmf of \((X_1, X_2, X_3)^T\). Make sure to specify the support of the distribution.

**Solution:** Make table. (The probability of each outcome is \( \frac{1}{12} \))

<table>
<thead>
<tr>
<th>Outcome</th>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_3 )</th>
<th>Let ( p(X_1, X_2, X_3) ) denote the joint pmf.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>( p(1, 1, 1) = \frac{1}{12} ) ( \checkmark )</td>
</tr>
<tr>
<td>1 2 3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( p(2, 1, 0) = \frac{1}{12} ) ( \checkmark )</td>
</tr>
<tr>
<td>1 3 1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>( p(2, 0, 1) = \frac{1}{12} ) ( \checkmark )</td>
</tr>
<tr>
<td>1 3 2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( p(2, 0, 1) = \frac{1}{12} ) ( \checkmark )</td>
</tr>
<tr>
<td>2 1 2</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>( p(1, 2, 0) = \frac{1}{12} ) ( \checkmark )</td>
</tr>
<tr>
<td>2 1 3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( p(0, 2, 1) = \frac{1}{12} ) ( \checkmark )</td>
</tr>
<tr>
<td>2 3 2</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>( p(0, 1, 2) = \frac{1}{12} ) ( \checkmark )</td>
</tr>
<tr>
<td>2 3 1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( p(1, 0, 2) = \frac{1}{12} ) ( \checkmark )</td>
</tr>
<tr>
<td>3 2 3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>( p(X_1, X_2, X_3) = 0 ) otherwise.</td>
</tr>
<tr>
<td>3 2 1</td>
<td>1</td>
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<td>3 1 2</td>
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<td>1</td>
<td>1</td>
<td>( p(X_1, X_2, X_3) = 0 ) otherwise.</td>
</tr>
</tbody>
</table>

(b) Are \( X_1 \) and \( X_2 \) independent? Prove your answer.

**Solution:** \( X_1 \) and \( X_2 \) are not independent. For example,

\[
\begin{align*}
p(X_1 = 1, X_2 = 2) = \frac{1}{12} \\
p(X_1 = 1) = \frac{3}{6} \quad \text{and} \quad p(X_2 = 2) = \frac{1}{6} \quad \text{and} \quad p(X_1 = 1) p(X_2 = 2) = \frac{3}{3} \cdot \frac{1}{6} = \frac{1}{6}
\end{align*}
\]

\( \therefore p(X_1 = 1, X_2 = 2) \neq p(X_1 = 1) p(X_2 = 2) \).
(2) Suppose \( X_1 \) and \( X_2 \) are jointly continuous with joint pdf
\[
f(x_1, x_2) = \begin{cases} 
\frac{3x_1^3}{x_2^4} e^{-x_1} & 0 < x_1, x_2 < \infty \\
0 & \text{otherwise}
\end{cases}
\]
Find \( \rho(X_1, X_2) \), the correlation coefficient between \( X_1 \) and \( X_2 \).

*Hint:* Integrate with respect to \( x_2 \) first.

**Sol:** First, note that the marginal pdf of \( X_1 \) is
\[
f_{x_1}(x_1) = \int_{0}^{\infty} \frac{3x_1^3}{x_2^4} e^{-x_1} \, dx_2 = 3x_1^3 \int_{x_1}^{\infty} e^{-x_2} \, dx_2
= 3x_1^3 e^{-x_1} \left( \frac{x_2^{-3}}{-3} \right)_{x_1}^{\infty}
= e^{-x_1}
\]
i.e., \( X_1 \sim \text{Exponential}(1) \). So \( E[X_1] = 1 \), \( \text{Var}(X_1) = 1 \).

Next, we have
\[
E[X_2^n] = \int_{0}^{\infty} \int_{0}^{\infty} x_2^n \frac{3x_1^3}{x_2^4} e^{-x_1} \, dx_1 \, dx_2 = \int_{0}^{\infty} 3x_1^3 \int_{0}^{\infty} x_2^{-4} \, dx_2 \, dx_1
= 3 \int_{x_1}^{\infty} x_2^{-4} \, dx_2 \left( \frac{x_2^{-n+1}}{n-1} \right)_{x_1}^{\infty}
= \frac{3}{3-n} \int_{0}^{\infty} x_1^n e^{-x_1} \, dx_1 \quad \text{for } n = 1 \text{ or } 2
= \frac{3 \Gamma(n+1)}{3-n} \int_{0}^{\infty} \frac{1}{\Gamma(n+1)} \frac{x_1^{n+1}}{\Gamma(n+1)} e^{-x_1} \, dx_1
= \frac{3n!}{3-n}
= \begin{cases} 
\frac{3}{2} & \text{if } n = 1 \\
6 & \text{if } n = 2
\end{cases}
\]
So \( E[X_2] = \frac{3}{2} \), \( E[X_2^2] = 6 \), \( \text{Var}(X_2) = 6 - \frac{9}{4} = \frac{15}{4} \).
\[
E(X_1 X_2) = \int_0^\infty \int_0^\infty x_1 x_2 \frac{3 x_1^3}{x_2^4} e^{-x_1} \, dx_2 \, dx_1
\]
\[
= \int_0^\infty 3 x_1^2 e^{-x_1} \left( \int_0^\infty x_2^{-3} \, dx_2 \right) \, dx_1
\]
\[
= \frac{3}{2} \int_0^\infty x_1^2 e^{-x_1} \, dx_1
\]
\[
= \frac{3}{2} \Gamma'(3) \int_0^\infty \Gamma(3) \frac{x_1^{3-1} e^{-x_1} \, dx_1}{\Gamma(3)}
\]
\[
= 3 \Gamma(3)
\]
Then \( \text{Cov}(X_1, X_2) = 3 - (1)(\frac{3}{2}) = \frac{3}{2} \) and
\[
\rho(X_1, X_2) = \frac{3/2}{\sqrt{\Gamma(1)(15/4)}} = \frac{3/2}{\sqrt{15}/2} = \frac{3}{\sqrt{15}} = \sqrt{\frac{3}{5}}
\]
3) Consider a square with vertices labelled 1, 2, 3, 4 (moving in a clockwise direction). A particle starts at vertex 1. On each move it chooses an adjacent vertex at random and moves to the chosen vertex. It repeatedly makes moves.

(a) Find the expected number of moves to reach vertex 2 for the first time, the expected number of moves to reach vertex 3 for the first time, and the expected number of moves to return to vertex 1. **Hint:** Define \(M_{ij}\) to be the expected number of moves to reach vertex \(j\) for the first time, starting from vertex \(i\) (if \(i = j\), \(M_{ii}\) is the expected number of moves to return to vertex \(i\) starting at vertex \(i\)).

**Sol.** Condition on the first move of the particle. We have

\[
M_{12} = (1) \frac{1}{2} + (1 + M_{13}) \frac{1}{2}
\]

\[
M_{13} = (1 + M_{23}) \frac{1}{2} + (1 + M_{12}) \frac{1}{2}
\]

By symmetry, \(M_{12} = M_{13}\), \(M_{23} = M_{12}\), \(M_{43} = M_{12}\). Then

(i) \(M_{12} = \frac{1}{2} + (1 + M_{13}) \frac{1}{2} = 1 + \frac{1}{2} M_{13}\)

(ii) \(M_{13} = (1 + M_{12}) \frac{1}{2} + (1 + M_{12}) \frac{1}{2} = 1 + M_{12}\)

Plugging (ii) into (i) gives \(M_{12} = 1 + \frac{1}{2} (1 + M_{12})\)

or \(\frac{1}{2} M_{12} = \frac{3}{2}\), or \(M_{12} = 3\)

Then \(M_{13} = 1 + M_{12} = 4\)

For \(M_{11}\), again conditioning on the first move, we have

\[
M_{11} = (1 + M_{12}) \frac{1}{2} + (1 + M_{41}) \frac{1}{2}
\]

By symmetry, \(M_{21} = M_{12}\) and \(M_{41} = M_{12}\). So, we get

\[
M_{11} = (1 + M_{12}) \frac{1}{2} + (1 + M_{12}) \frac{1}{2} = 1 + M_{12} = 1 + 3 = 4 .
\]
(b) What is the expected number of moves until vertices 2 and 3 have both been reached?

Hint: Define $M_{ijk}$, $i,j,k$ distinct, to be the expected number of moves until both vertices $j$ and $k$ have been reached starting from vertex $i$.

So, we want $M_{1,23}$. Conditioning on first move, we have

$$M_{1,23} = (1 + M_{23}) \frac{1}{2} + (1 + M_{4,23}) \frac{1}{2}$$

By symmetry, $M_{4,23} = M_{1,23}$ and $M_{23} = M_{12} = 3$. So

$$M_{1,23} = (1 + 3) \frac{1}{2} + (1 + M_{1,23}) \frac{1}{2}$$

or

$$\frac{1}{2} M_{1,23} = 2 + \frac{1}{2} \quad \text{or} \quad M_{1,23} = 5$$

So, $M_{1,23} = 5$. The expected number of moves until vertices 2 and 3 have both been reached is 5.