Student Number

Queen's University
Department of Mathematics and Statistics

MTHE/STAT 353

Final Examination April 16, 2014
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- "Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written."
- "The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer."
- Formulas and tables are attached.
- An 8.5×11 inch sheet of notes (both sides) is permitted.
- Simple calculators are permitted (Casio 991, red, blue, or gold sticker). HOWEVER, do reasonable simplifications.
- Write the answers in the space provided, continue on the backs of pages if needed.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.
- Marks per part question are shown in brackets at the right margin.

Marks: Please do not write in the space below.

Problem 1 [10]	Problem 4 [10]
Problem 2 [10]	Problem 5 [10]
Problem 3 [10]	Problem 6 [10]

Total: [60]

1. Let X and Y be independent and identically distributed uniform random variables on the interval $(0, 1)$. Define $U = \frac{1}{2}(X + Y)$ to be the average and define $V = X$.

(a) Find the joint probability density function of (U, V) and draw the sample space of (U, V). (Be careful when determining the sample space of (U, V) – it will affect your answer in part(b).) [6]

(b) Find the marginal probability density function of U. [4]

2. A mouse is placed at the starting point in a maze. There are three possible directions the mouse can travel from the starting point - one direction to the left, one to the right and one straight ahead. If the mouse travels to the left, then on average it will wander the maze for 2 minutes and then return to the starting point. If the mouse travels straight ahead then on average it will wander the maze for 1 minute and then find the exit to the maze. If the mouse travels to the right then on average it will wander the maze for 5 minutes and then return to the starting point.

(a) Under the hypothesis that whenever the mouse is at the starting point it chooses one of the three possible directions at random and starts travelling in the chosen direction, find the expected amount of time the mouse spends in the maze before exiting. [5]

(b) Under the hypothesis that the mouse learns, so that whenever it is at the starting point it chooses at random one of the directions it has not tried before, find the expected amount of time the mouse spends in the maze before exiting. [5]

3. Suppose that k balls are randomly placed into n boxes (which are initially empty). For $i = 1, \ldots, n$, let

$$X_i = \begin{cases} 1 & \text{if box } i \text{ is empty} \\ 0 & \text{if box } i \text{ has one or more balls.} \end{cases}$$

(a) Find the expected number of empty boxes. [2]

(b) Find $\mathrm{Cov}(X_1, X_2)$. [5]

(c) Show that X_1 and X_2 are negatively correlated. [3]

4. (a) Suppose that $\{X_n\}$ is a sequence of zero-mean random variables and X is a zero-mean random variable, and suppose that $E[(X_n - X)^2] \leq C/n^p$ for every n, for some constants C and $p > 1$. Show that $X_n \to X$ almost surely. [4]

(b) Suppose that $\{X_n\}$ is a sequence of nonnegative random variables. Show that $E[X_n] \to 0$ as $n \to \infty$ implies that $X_n \to 0$ in probability, but that the converse is false in general. [6]

5. (a) Give explicitly a sequence of random variables $\{X_n\}$ (i.e., give the probability space Ω, the probability measure P on Ω, and the random variables (functions) from Ω to $\mathbb{R}$) such that $X_n \to 0$ almost surely but it does not hold that $\sum_{n=1}^{\infty} P(|X_n - 0| > \epsilon) < \infty$ for any $\epsilon > 0$. [5]

(b) Suppose that $\{X_n\}$ and $\{Y_n\}$ are sequences of random variables and X and Y are random variables such that $X_n \to X$ in distribution and $Y_n \to Y$ in distribution. Give an example where it is *not* true that $X_n + Y_n$ converges to $X + Y$ in distribution. *Hint*: Consider $Y = -X$. [5]

6. Let $\{X_n\}$ be a sequence of independent and identically distributed random variables with mean μ and variance σ^2. Let g be a strictly monotone function (strictly increasing or strictly decreasing) and differentiable. Find directly using the central limit theorem (*not* the delta method), the limit of $P(g(\overline{X}_n) \le g(\mu)+t/\sqrt{n})$ as $n \to \infty$, where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $t > 0$ is a fixed real number. *Notes:* (1) You may assume that if $\{Y_n\}$ is a sequence of random variables and Y is another random variables such that $Y_n \to Y$ in distribution and if $\{y_n\}$ is a sequence of numbers such that $y_n \to y$, where the distribution function of Y is continuous at y, then $P(Y_n \le y_n) \to P(Y \le y)$; (2) Recall from calculus that if $y = g(x)$, where g is monotone and differentiable, then $\frac{d}{dy}g^{-1}(y) = [\frac{d}{dx}g(x)]^{-1}$. [10]

Formula Sheet

Special Distributions

Uniform on the interval $(0, 1)$:

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases} \qquad E[X] = \frac{1}{2}, \quad \mathrm{Var}(X) = \frac{1}{12}.$$