Student Number

Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

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Instructor: G. Takahara

- "Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written."
- "The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer."
- Formulas and tables are attached.
- An 8.5×11 inch sheet of notes (both sides) is permitted.
- Simple calculators are permitted (Casio 991, red, blue, or gold sticker). HOWEVER, do reasonable simplifications.
- Write the answers in the space provided, continue on the backs of pages if needed.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.
- Marks per part question are shown in brackets at the right margin.

Marks: Please do not write in the space below.

Problem 1 [10]	Problem 4 [10]
Problem 2 [10]	Problem 5 [10]
Problem 3 [10]	Problem 6 [10]

Total: [60] ____________

1. Let X and Y be independent and identically distributed uniform random variables on the interval $(0,1)$. Define $U = \frac{1}{2}(X+Y)$ to be the average and define $V = X$.

(a) Find the joint probability density function of (U,V) and draw the sample space of (U,V). (Be careful when determining the sample space of (U,V) – it will affect your answer in part(b).) [6]

<u>Solution:</u> The inverse transformation is $X = V$ and $Y = 2U - V$. The matrix of partial derivatives of the inverse transformation is

$$\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$

with determinant -2. The joint pdf of (X,Y) is

$$f_{XY}(x,y) = \begin{cases} 1 & \text{for } 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

So the joint pdf of (U,V) is

$$f_{UV}(u,v) = \begin{cases} 2 & \text{for } (u,v) \in S_{UV} \\ 0 & \text{otherwise,} \end{cases}$$

where S_{UV} is the sample space of (U,V), determined by the constraints $0 \le v \le 1$ and $0 \le 2u - v \le 1$, or $0 \le v \le 1$ and $v/2 \le u \le (v+1)/2$. The sample space of (U,V) is plotted in Figure 1.

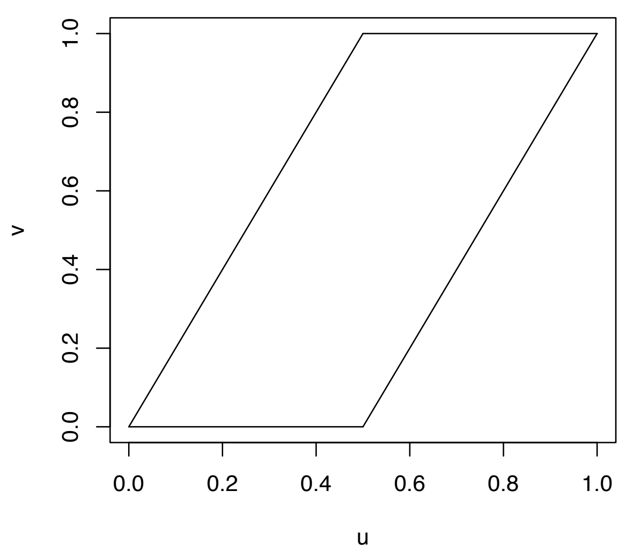

Figure 1: Sample space of (U,V).

(b) Find the marginal probability density function of U. [4]

<u>Solution:</u> To get the marginal pdf of U we integrate the joint pdf $f_{UV}(u, v)$ over the variable v from $-\infty$ to ∞. From Figure 1 we see that the limits of integration are different depending on whether $u \in [0, .5]$ or $u \in (.5, 1]$. For $u \in [0, .5]$ we have

$$f_U(u) = \int_0^{2u} (2)dv = 4u.$$

For $u \in (.5, 1]$ we have

$$f_U(u) = \int_{2u-1}^{1} (2)dv = 2(1 - (2u - 1)) = 4(1 - u).$$

Clearly, for $u \notin [0, 1]$ we have $f_U(u) = 0$. To summarize,

$$f_U(u) = \begin{cases} 4u & \text{for } u \in [0, .5] \\ 4(1 - u) & \text{for } u \in (.5, 1] \\ 0 & \text{otherwise.} \end{cases}$$

2. A mouse is placed at the starting point in a maze. There are three possible directions the mouse can travel from the starting point - one direction to the left, one to the right and one straight ahead. If the mouse travels to the left, then on average it will wander the maze for 2 minutes and then return to the starting point. If the mouse travels straight ahead then on average it will wander the maze for 1 minute and then find the exit to the maze. If the mouse travels to the right then on average it will wander the maze for 5 minutes and then return to the starting point.

(a) Under the hypothesis that whenever the mouse is at the starting point it chooses one of the three possible directions at random and starts travelling in the chosen direction, find the expected amount of time the mouse spends in the maze before exiting. [5]

Solution: Letting T denote the amount of time the mouse spends in the maze before exiting, and conditioning on the first move of the mouse, we have

$$\begin{aligned} E[T] &= \frac{1}{3}\Big(E[T \mid \text{left}] + E[T \mid \text{right}] + E[T \mid \text{straight ahead}]\Big) \\ &= \frac{1}{3}\Big(2 + E[T] + 1 + 5 + E[T]\Big) = \frac{1}{3}(8 + 2E[T]). \end{aligned}$$

Therefore, $\frac{1}{3}E[T] = \frac{8}{3}$, or $E[T] = 8$ minutes.

(b) Under the hypothesis that the mouse learns, so that whenever it is at the starting point it chooses at random one of the directions it has not tried before, find the expected amount of time the mouse spends in the maze before exiting. [5]

Solution: We begin as in part(a), but now

$$E[T \mid \text{left}] = 2 + \frac{1}{2}(1 + 5 + 1) = \frac{11}{2}$$

and

$$E[T \mid \text{right}] = 5 + \frac{1}{2}(1 + 2 + 1) = 7.$$

Therefore,

$$E[T] = \frac{1}{3}\left(\frac{11}{2} + 1 + 7\right) = \frac{9}{2} = 4.5 \text{ minutes.}$$

3. Suppose that k balls are randomly placed into n boxes (which are initially empty). For $i = 1, \dots, n$, let

$$X_i = \begin{cases} 1 & \text{if box } i \text{ is empty} \\ 0 & \text{if box } i \text{ has one or more balls.} \end{cases}$$

(a) Find the expected number of empty boxes. [2]

Solution: We have $P(X_i = 1) = (\frac{n-1}{n})^k$ since each ball independently will not end up in box i with probability $\frac{n-1}{n}$. The number of empty boxes is $X_1 + \ldots + X_n$ so the expected number of empty boxes is $E[X_1] + \ldots + E[X_n] = n(\frac{n-1}{n})^k$.

(b) Find $\mathrm{Cov}(X_1, X_2)$. [5]

Solution: $E[X_1] = E[X_2] = (\frac{n-1}{n})^k$ were computed in part(a). To compute $\mathrm{Cov}(X_1, X_2)$ we further need $E[X_1 X_2]$. This is given by

$$E[X_1 X_2] = P(X_1 = 1, X_2 = 1) = \left(\frac{n-2}{n}\right)^k,$$

since each of the k balls independently does not end up in either box 1 or box 2 with probability $\frac{n-2}{n}$. Then

$$\mathrm{Cov}(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2] = \left(\frac{n-2}{n}\right)^k - \left(\frac{n-1}{n}\right)^{2k}.$$

(c) Show that X_1 and X_2 are negatively correlated. [3]

Solution: To show that X_1 and X_2 are negatively correlated it suffices to show that the covariance computed in part(b) is negative. For this it is sufficient to show that

$$\begin{aligned} \frac{n-2}{n} < \left(\frac{n-1}{n}\right)^2 \quad &\Leftrightarrow \quad n(n-2) < (n-1)^2 \\ &\Leftrightarrow \quad n^2 - 2n < n^2 - 2n + 1 \end{aligned}$$

which is true for any positive n.

4. (a) Suppose that $\{X_n\}$ is a sequence of zero-mean random variables and X is a zero-mean random variable, and suppose that $E[(X_n - X)^2] \le C/n^p$ for every n, for some constants C and $p > 1$. Show that $X_n \to X$ almost surely. [4]

Solution: Let $\epsilon > 0$ be given. Since $E[X_n - X] = 0$ we have by Chebyshev's inequality that

$$P(|X_n - X| > \epsilon) \le \frac{\mathrm{Var}(X_n - X)}{\epsilon^2} = \frac{E[(X_n - X)^2]}{\epsilon^2} \le \frac{C}{\epsilon^2 n^p}.$$

Therefore,

$$\sum_{n=1}^{\infty} P(|X_n - X| > \epsilon) \le \sum_{n=1}^{\infty} \frac{C}{\epsilon^2 n^p} < \infty$$

if $p > 1$. By a sufficient condition from class this implies that $X_n \to X$ almost surely.

(b) Suppose that $\{X_n\}$ is a sequence of nonnegative random variables. Show that $E[X_n] \to 0$ as $n \to \infty$ implies that $X_n \to 0$ in probability, but that the converse is false in general. [6]

Solution: Suppose that $E[X_n] \to 0$. Let $\epsilon > 0$ be given. Then by Markov's inequality

$$P(|X_n - 0| > \epsilon) \le \frac{E[X_n]}{\epsilon} \to 0$$

as $n \to \infty$. Hence, $X_n \to 0$ in probability. To prove that the reverse implication is false in general we give a counterexample. Let the distribution of X_n be given by

$$X_n = \begin{cases} 0 & \text{with probability } 1 - \frac{1}{n} \\ n & \text{with probability } \frac{1}{n} \end{cases}$$

Then if $\epsilon > 0$ is given $P(|X_n - 0| > \epsilon) = P(X_n = n) = \frac{1}{n} \to 0$ as $n \to \infty$, so $X_n \to 0$ in probability. However, $E[X_n] = n(\frac{1}{n}) = 1$ for all n, which does not converge to 0.

5. (a) Give explicitly a sequence of random variables $\{X_n\}$ (i.e., give the probability space Ω, the probability measure P on Ω, and the random variables (functions) from Ω to $\mathbb{R}$) such that $X_n \to 0$ almost surely but it does not hold that $\sum_{n=1}^{\infty} P(|X_n - 0| > \epsilon) < \infty$ for any $\epsilon > 0$. [5]

<u>Solution:</u> Let $\Omega = [0,1]$ and P the uniform distribution on $[0,1]$. Define X_n as follows:

$$X_n(\omega) = \begin{cases} 0 & \text{if } \omega \in (1/n, 1] \\ 1 & \text{if } \omega \in [0, 1/n] \end{cases}$$

Then for $\omega \in (0,1]$, $X_n(\omega) = 0$ for all $n > 1/\omega$ so $X_n(\omega) \to 0$ as $n \to \infty$. Since $P((0,1]) = 1$, we have that $X_n \to 0$ almost surely. If $\epsilon > 0$ is given then $P(|X_n - 0| > \epsilon) = P(X_n = 1) = \frac{1}{n}$. Therefore,

$$\sum_{n=1}^{\infty} P(|X_n - 0| > \epsilon) = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is a divergent sum.

(b) Suppose that $\{X_n\}$ and $\{Y_n\}$ are sequences of random variables and X and Y are random variables such that $X_n \to X$ in distribution and $Y_n \to Y$ in distribution. Give an example where it is *not* true that $X_n + Y_n$ converges to $X + Y$ in distribution. *Hint*: Consider $Y = -X$. [5]

<u>Solution:</u> Let W and Z be independent $N(0,1)$ random variables. Let $X_n = W$ for all n and $Y_n = Z$ for all n. Let $X = W$ and $Y = -W$. It is easy to see that X and Y are both $N(0,1)$, and it is clear that $X_n \to X$ in distribution and $Y_n \to Y$ in distribution. But $X + Y = 0$ whereas $X_n + Y_n$ is distributed as $N(0,2)$ for all n. Therefore, it is not true that $X_n + Y_n$ converges to $X + Y$ in distribution.

6. Let $\{X_n\}$ be a sequence of independent and identically distributed random variables with mean μ and variance σ^2. Let g be a strictly monotone function (strictly increasing or strictly decreasing) and differentiable. Find directly using the central limit theorem (*not* the delta method), the limit of $P(g(\overline{X}_n) \le g(\mu)+t/\sqrt{n})$ as $n \to \infty$, where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $t > 0$ is a fixed real number. *Notes:* (1) You may assume that if $\{Y_n\}$ is a sequence of random variables and Y is another random variables such that $Y_n \to Y$ in distribution and if $\{y_n\}$ is a sequence of numbers such that $y_n \to y$, where the distribution function of Y is continuous at y, then $P(Y_n \le y_n) \to P(Y \le y)$; (2) Recall from calculus that if $y = g(x)$, where g is monotone and differentiable, then $\frac{d}{dy}g^{-1}(y) = [\frac{d}{dx}g(x)]^{-1}$. [10]

<u>Solution:</u> We first write

$$\begin{aligned} P(g(\overline{X}_n \le g(\mu) + t/\sqrt{n}) &= P(\overline{X}_n \le g^{-1}(g(\mu) + t/\sqrt{n})) \\ &= P\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \le \frac{g^{-1}(g(\mu) + t/\sqrt{n}) - \mu}{\sigma/\sqrt{n}}\right) \\ &= P\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \le y_n\right), \end{aligned}$$

where

$$y_n = \frac{g^{-1}(g(\mu) + t/\sqrt{n}) - \mu}{\sigma/\sqrt{n}}.$$

Then if $y_n \to y$ as $n \to \infty$, by the central limit theorem and Note 1, the probability in question will converge to $\Phi(y)$, where $\Phi(\cdot)$ is the standard normal distribution function. It remains to find $\lim_{n\to\infty} y_n$. For a more convenient variable we can set $h = 1/\sqrt{n}$, then

$$\lim_{n\to\infty} y_n = \lim_{h\downarrow 0} \frac{g^{-1}(g(\mu) + th) - \mu}{\sigma h}. \tag{1}$$

Both numerator and denominator above go to 0 as $h \downarrow 0$ so we apply L'Hôspital's rule. For the denominator, $\frac{d}{dh}\sigma h = \sigma$. For the numerator, with $y = g(\mu)+th$ and $x = g^{-1}(g(\mu)+th)$ and applying Note 2, we have (first applying the chain rule)

$$\frac{d}{dh}g^{-1}(g(\mu) + th) - \mu = \frac{d}{dy}g^{-1}(y)t = \frac{t}{\frac{d}{dx}g(x)} = \frac{t}{\frac{d}{dx}g(g^{-1}(g(\mu) + th))}.$$

As $h \downarrow 0$ the denominator in the final expression above goes to $g'(\mu)$. Going back to Eq.(1) we have that $\lim_{n\to\infty} y_n = \frac{t}{\sigma g'(\mu)}$, and so

$$P(g(\overline{X}_n \le g(\mu) + t/\sqrt{n}) \to \Phi\left(\frac{t}{\sigma g'(\mu)}\right)$$

as $n \to \infty$ (which of course is easily obtained if one resorted to applying the Delta method).

Formula Sheet

Special Distributions

Uniform on the interval $(0,1)$:

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases} \qquad E[X] = \frac{1}{2}, \quad \mathrm{Var}(X) = \frac{1}{12}.$$