Student Number

Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

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- "Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written."
- "The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer."
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- Formulas and tables are attached.
- An 8.5×11 inch sheet of notes (both sides) is permitted. Simple calculators are permitted (Casio 991, red, blue, or gold sticker). HOWEVER, do reasonable simplifications.
- Write the answers in the space provided, continue on the backs of pages if needed.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.
- Marks per part question are shown in brackets at the right margin.

Marks: Please do not write in the space below.

Problem 1 [10]	Problem 4 [10]
Problem 2 [10]	Problem 5 [10]
Problem 3 [10]	Problem 6 [10]

Total: [60]

1. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables, each with a Uniform distribution on the interval $(0, 1)$. Let $X = \min(X_1, \ldots, X_n)$ and $Y = \max(X_1, \ldots, X_n)$.

(a) Find $P(X < \frac{1}{2} < Y)$. [6]

(b) Find $E[X^3]$. [4]

2. An urn contains 10 red balls, 10 blue balls, and 10 green balls. Ten balls are drawn at random without replacement. For each green ball in the sample, it is replaced by a red ball with probability 1/2 and by a blue ball with probability 1/2, independently for each green ball in the sample. Let X denote the total number of red balls and Y the total number of blue balls in the sample after the green balls are replaced. Find $\mathrm{Cov}(X, Y)$. [10]

3. Let $M > 1$ be a given positive constant. Let $X, X_1, X_2, \ldots$ be nonnegative random variables.

(a) Suppose that $E[X^n] \leq M$ for all $n \geq 1$. Show that $P(X > 1 + \epsilon) = 0$ for any $\epsilon > 0$. [5]

(b) Suppose that $E[X_n^n] \leq M$ for all n. Show by counterexample that $P(X_n > 1+\epsilon) = 0$ for any $\epsilon > 0$ is not necessarily true. [5]

4. Let X have a Gamma$(3, 3)$ distribution. Conditional on $X = x$ let Z have a normal distribution with mean x and variance 2. Finally, let $Y = e^Z$. Find $E[Y]$ and Var(Y). [10]

5. Let $X_1, X_2, \ldots$ be a sequence of independent random variables, where X_n is Exponentially distributed with mean $1/\ln n^p$, where $p > 0$ is a fixed constant.

(a) Does X_n converge in probability to a limiting random variable as $n \to \infty$? If so, give the limit and prove the convergence. If not, prove it. [4]

(b) Does X_n converge almost surely to a limiting random variable as $n \to \infty$? If so, give the limit and prove the convergence. If not, prove it. [6]

6. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables, each with a Poisson distribution with mean 1. Let $S_n = X_1 + \ldots + X_n$ for $n \geq 1$ and let $M_n(t)$ be the moment generating function of S_n.

(a) Find the smallest n such that $P(M_n(S_n) > 1) \geq .99$ using the exact probability. [6]

(b) Find the smallest n such that $P(M_n(S_n) > 1) \geq .99$ using the central limit theorem. [4]

Formula Sheet

Special Distributions

Beta distribution with parameters $\alpha > 0$ and $\beta > 0$:

$$\begin{aligned} f(x) &= \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases} \\ E[X] &= \frac{\alpha}{\alpha+\beta}, \quad \mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \end{aligned}$$

$\alpha = 1$ and $\beta = 1$ gives the Uniform distribution on $(0, 1)$.

Gamma distribution with parameters $r > 0$ and $\lambda > 0$:

$$\begin{aligned} f(x) &= \begin{cases} \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases} \qquad E[X] = \frac{r}{\lambda}, \quad \mathrm{Var}(X) = \frac{r}{\lambda^2} \\ M_X(t) &= \left(\frac{\lambda}{\lambda - t}\right)^r \quad \text{for } t < \lambda. \end{aligned}$$

$r = 1$ gives the Exponential distribution with mean $1/\lambda$.

Normal distribution with mean μ and variance $\sigma^2 > 0$:

$$\begin{aligned} f(x) &= \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} & \text{if } x \in \mathbb{R} \\ 0 & \text{otherwise.} \end{cases} \qquad E[X] = \mu, \quad \mathrm{Var}(X) = \sigma^2 \\ M_X(t) &= e^{\mu t + t^2\sigma^2/2} \quad \text{for } t \in \mathbb{R}. \end{aligned}$$

$\mu = 0$ and $\sigma^2 = 1$ gives the standard normal distribution.

Poisson distribution with mean $\lambda > 0$:

$$\begin{aligned} f(k) &= \begin{cases} \frac{\lambda^k}{k!} e^{-\lambda} & \text{if } k = 0, 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \qquad E[X] = \lambda, \quad \mathrm{Var}(X) = \lambda \\ M_X(t) &= \exp\{\lambda(e^t - 1)\} \quad \text{for } t \in \mathbb{R}. \end{aligned}$$

The distribution function, $\Phi(z)$, of a standard normal random variable

Note: $\Phi(-z) = 1 - \Phi(z)$ for any $z \in \mathbb{R}$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998