Student Number

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Department of Mathematics and Statistics

MTHE/STAT 353

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- "Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written."
- "The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer."
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- Formulas and tables are attached.
- An 8.5 × 11 inch sheet of notes (both sides) is permitted. Simple calculators are permitted (Casio 991, red, blue, or gold sticker). HOWEVER, do reasonable simplifications.
- Write the answers in the space provided, continue on the backs of pages if needed.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.
- Marks per part question are shown in brackets at the right margin.

Marks: Please do not write in the space below.

Problem 1 [10]	Problem 4 [10]
Problem 2 [10]	Problem 5 [10]
Problem 3 [10]	Problem 6 [10]

Total: [60]

1. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables, each with a Uniform distribution on the interval $(0,1)$. Let $X = \min(X_1, \ldots, X_n)$ and $Y = \max(X_1, \ldots, X_n)$.

(a) Find $P(X < \frac{1}{2} < Y)$. [6]

Solution: The joint pdf of (X, Y) is

$$f(x,y) = \begin{cases} n(n-1)(y-x)^{n-2} & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} P(X < \frac{1}{2} < Y) &= \int_{1/2}^{1} \int_{0}^{1/2} n(n-1)(y-x)^{n-2} dx dy \\ &= \int_{1/2}^{1} n \left[-(y-x)^{n-1} \Big|_0^{1/2} \right] dy \\ &= \int_{1/2}^{1} n y^{n-1} dy - \int_{1/2}^{1} n \left(y - \frac{1}{2} \right)^{n-1} dy \\ &= y^n \Big|_{1/2}^{1} - \left(y - \frac{1}{2} \right)^n \Big|_{1/2}^{1} \\ &= 1 - \frac{1}{2^n} - \frac{1}{2^n} \\ &= 1 - \frac{1}{2^{n-1}}. \end{aligned}$$

(b) Find $E[X^3]$. [4]

Solution: The marginal pdf of X is

$$f_X(x) = \begin{cases} n(1-x)^{n-1} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$E[X^3] = \int_0^1 x^3 n(1-x)^{n-1} dx = n \int_0^1 x^3 (1-x)^{n-1} dx.$$

The integral on the right is $B(4, n)$, where $B(\cdot, \cdot)$ is the beta function. So

$$E[X^3] = nB(4, n) = n\frac{\Gamma(4)\Gamma(n)}{\Gamma(n+4)} = \frac{6n}{(n+3)(n+2)(n+1)n} = \frac{6}{(n+3)(n+2)(n+1)}.$$

2. An urn contains 10 red balls, 10 blue balls, and 10 green balls. Ten balls are drawn at random without replacement. For each green ball in the sample, it is replaced by a red ball with probability 1/2 and by a blue ball with probability 1/2, independently for each green ball in the sample. Let X denote the total number of red balls and Y the total number of blue balls in the sample after the green balls are replaced. Find $\mathrm{Cov}(X, Y)$. [10]

Solution: Write $X = X_1 + \ldots + X_{10}$ and $Y = Y_1 + \ldots + Y_{10}$, where X_i and Y_i are the indicators that the ith ball drawn ends up being red or blue, respectively. Then

$$\mathrm{Cov}(X, Y) = \mathrm{Cov}\Big(\sum_{i=1}^{10} X_i, \sum_{j=1}^{10} Y_j\Big) = \sum_{i=1}^{10}\sum_{j=1}^{10} \mathrm{Cov}(X_i, Y_j) = 10\mathrm{Cov}(X_1, Y_1) + 90\mathrm{Cov}(X_1, Y_2),$$

where the last equality follows because, by symmetry, $\mathrm{Cov}(X_i, Y_i)$ is the same for all i and $\mathrm{Cov}(X_i, Y_j)$ is the same for all $i \neq j$. We have

$$\begin{aligned}\mathrm{Cov}(X_1, Y_1) &= E[X_1 Y_1] - E[X_1]E[Y_1] \\ &= P(X_1 = 1, Y_1 = 1) - P(X_1 = 1)P(Y_1 = 1) = -P(X_1 = 1)^2,\end{aligned}$$

where the last equality follows because, again by symmetry, $P(X_1 = 1) = P(Y_1 = 1)$. Similarly, we also have $P(X_1 = 1) = P(Y_2 = 1)$ and so

$$\mathrm{Cov}(X_1, Y_2) = P(X_1 = 1, Y_2 = 1) - P(X_1 = 1)^2.$$

So it boils down to computing $P(X_1 = 1)$ and $P(X_1 = 1, Y_2 = 1)$. The event $\{X_1 = 1\}$ occurs if and only if either the first ball is red or the first ball is green and it is replaced by a red ball. The probability of this is $\frac{10}{30} + \frac{10}{30} \times \frac{1}{2} = \frac{1}{2}$. The event $\{X_1 = 1, Y_2 = 1\}$ occurs if and only if (i) the first ball is red and the second ball is blue; (ii) the first ball is green and is replaced by a red ball and the second ball is blue; (iii) the first ball is red and the second ball is green and is replaced by a blue ball; or (iv) the first ball is green and is replaced by a red ball and the second ball is green and is replaced by a blue ball. The probability of the union of (i) to (iv) is (note (ii) and (iii) have the same probability)

$$\begin{aligned}P(X_1 = 1, Y_2 = 1) &= \left(\frac{10}{30}\right)\left(\frac{10}{29}\right) + 2\left(\frac{10}{30}\right)\left(\frac{1}{2}\right)\left(\frac{10}{29}\right) + \left(\frac{10}{30}\right)\left(\frac{1}{2}\right)\left(\frac{9}{29}\right)\left(\frac{1}{2}\right) \\ &= \frac{20}{87} + \frac{3}{116} = \frac{89}{348}.\end{aligned}$$

Then

$$\mathrm{Cov}(X, Y) = -\frac{10}{4} + \frac{90 \times 89}{348} - \frac{90}{4} = \frac{90 \times 89 - 100 \times 87}{348} = -\frac{690}{348} = -\frac{115}{58} \approx -1.983.$$

3. Let $M > 1$ be a given positive constant. Let $X, X_1, X_2, \ldots$ be nonnegative random variables.

(a) Suppose that $E[X^n] \le M$ for all $n \ge 1$. Show that $P(X > 1+\epsilon) = 0$ for any $\epsilon > 0$. [5]

Solution: Let $\epsilon > 0$ be given. By Markov's inequality

$$P(X > 1+\epsilon) = P(X^n > (1+\epsilon)^n) \le \frac{E[X^n]}{(1+\epsilon)^n} \le \frac{M}{(1+\epsilon)^n},$$

which holds for all $n \ge 1$. Since the right hand side can be made arbitrarily small the only way the inequality can hold is if $P(X > 1+\epsilon) = 0$.

(b) Suppose that $E[X_n^n] \le M$ for all n. Show by counterexample that $P(X_n > 1+\epsilon) = 0$ for any $\epsilon > 0$ is not necessarily true. [5]

Solution: Let

$$X_n = \begin{cases} 0 & \text{with probability } 1 - \frac{1}{M^{n-1}} \\ M & \text{with probability } \frac{1}{M^{n-1}} \end{cases}$$

Then $E[X_n^n] = M^n \times \frac{1}{M^{n-1}} = M$ for all n but for $\epsilon \in (0,1)$ we have $P(X_n > 1+\epsilon) = P(X_n = M) = \frac{1}{M^{n-1}} > 0$.

4. Let X have a Gamma$(3,3)$ distribution. Conditional on $X=x$ let Z have a normal distribution with mean x and variance 2. Finally, let $Y=e^Z$. Find $E[Y]$ and Var(Y). [10]

Solution: Conditioned on $X=x$, $E[Y]$ is the moment generating function of a $N(x,2)$ distribution evaluated at 1. Letting $M_x(\cdot)$ denote this mgf, we have $M_x(t)=e^{xt+t^2}$, and so $E[Y \mid X=x]=M_x(1)=e^{x+1}$. Similarly, $E[Y^2 \mid X=x]=E[e^{2Z} \mid X=x]=M_x(2)=e^{2x+4}$. Then by the law of total expectation, $E[Y]=E[e^{X+1}]=eM_X(1)$ and $E[Y^2]=E[e^{2X+4}]=e^4M_X(2)$, where $M_X(\cdot)$ is the moment generating function of X. Since X has a Gamma$(3,3)$ distribution we have $M_X(t)=\left(\frac{3}{3-t}\right)^3$, and so

$$E[Y]=e\left(\frac{3}{2}\right)^3=\frac{27e}{8}\approx 9.174 \quad \text{and} \quad E[Y^2]=27e^4.$$

Then

$$\mathrm{Var}(Y)=E[Y^2]-E[Y]^2=27e^4-\left(\frac{27}{8}\right)^2e^2\approx 1390.$$

5. Let $X_1, X_2, \ldots$ be a sequence of independent random variables, where X_n is Exponentially distributed with mean $1/\ln n^p$, where $p > 0$ is a fixed constant.

(a) Does X_n converge in probability to a limiting random variable as $n \to \infty$? If so, give the limit and prove the convergence. If not, prove it. [4]

Solution: Since X_n is positive and $E[X_n]$ is converging to 0, if X_n converges in probability it should be to the constant 0. To prove this, let $\epsilon > 0$ be given and compute

$$P(|X_n - 0| > \epsilon) = P(X_n > \epsilon) = e^{-\epsilon \ln n^p} = \frac{1}{n^{\epsilon p}} \to 0$$

as $n \to \infty$. This proves that $X_n \to_P 0$.

(b) Does X_n converge almost surely to a limiting random variable as $n \to \infty$? If so, give the limit and prove the convergence. If not, prove it. [6]

Solution: If X_n converges almost surely it must be to the constant 0 since in part(a) it was shown that X_n converges to 0 in probability, and almost sure convergence implies convergence in probability. Since $\sum_n \frac{1}{n^{\epsilon p}}$ diverges for ϵ small enough the sufficient condition for almost sure convergence is not satisfied, so it is possible that X_n does not converge almost surely. To check this we compute

$$\prod_{n=m}^{\infty} P(X_n \le \epsilon) = \lim_{M\to\infty} \prod_{n=m}^{M} P(X_n \le \epsilon) = \lim_{M\to\infty} \prod_{n=m}^{M} \left(1 - \frac{1}{n^{\epsilon p}}\right).$$

In particular, choosing $\epsilon = \frac{1}{p}$ we have

$$\prod_{n=m}^{\infty} P(X_n \le 1/p) = \lim_{M\to\infty} \prod_{n=m}^{M} \left(1 - \frac{1}{n}\right) = \lim_{M\to\infty} \frac{m-1}{M} = 0.$$

Thus, $P(\cup_{n=m}^{\infty}\{X_n > 1/p\}) = 1 - P(\cap_{n=m}^{\infty}\{X_n \le 1/p\}) = 1 - \prod_{n=m}^{\infty} P(X_n \le 1/p) = 1 - 0 = 1$ for every m, and so $P(\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} \{X_n > 1/p\}) = 1$, since a countable intersection of events of probability 1 has probability 1. If $\omega \in \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} \{X_n > 1/p\}$ then $X_n(\omega) > 1/p$ for infinitely many n, so $X_n(\omega)$ does not converge to 0. So with probability 1, X_n does not converge to 0 and so X_n does not converge almost surely to any limit.

6. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables, each with a Poisson distribution with mean 1. Let $S_n = X_1 + \ldots + X_n$ for $n \geq 1$ and let $M_n(t)$ be the moment generating function of S_n.

(a) Find the smallest n such that $P(M_n(S_n) > 1) \geq .99$ using the exact probability. [6]

Solution: First, the common moment generating function of the X_i is

$$M(t) = E[e^{tX_i}] = \sum_{k=0}^{\infty} e^{tk} \frac{1}{k!} e^{-1} = e^{-1} \sum_{k=0}^{\infty} \frac{(e^t)^k}{k!} = e^{-1} e^{e^t} = e^{e^t - 1}.$$

Then $M_n(t) = M(t)^n = e^{n(e^t-1)}$ and $M_n(S_n) = e^{n(e^{S_n}-1)}$. So

$$P(M_n(S_n) > 1) = P(e^{n(e^{S_n}-1)} > 1) = P(n(e^{S_n} - 1) > 0) = P(e^{S_n} > 1) = P(S_n > 0).$$

The exact distribution of S_n is Poisson(n), so $P(S_n > 0) = 1 - P(S_n = 0) = 1 - e^{-n}$. Setting this to .99 gives $e^{-n} = .01$, or $n = \ln 100 = 4.605$. So $n = 5$ is the smallest n.

(b) Find the smallest n such that $P(M_n(S_n) > 1) \geq .99$ using the central limit theorem. [4]

Solution: By the central limit theorem, the distribution of $(S_n - n)/\sqrt{n}$ is approximated by a $N(0,1)$ distribution, so that (from part(a)),

$$P(M_n(S_n) > 1) = P(S_n > 0) = P((S_n - n)/\sqrt{n} > -\sqrt{n}) \approx 1 - \Phi(-\sqrt{n}),$$

where $\Phi(\cdot)$ is the standard normal cdf. Setting this to .99 gives $\Phi(-\sqrt{n}) = .01$, or (from the table) $-\sqrt{n} = -2.33$, or $n = 5.429$. So $n = 6$ is the smallest n according to this approximation.

Formula Sheet

Special Distributions

Beta distribution with parameters $\alpha > 0$ and $\beta > 0$:

$$f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X] = \frac{\alpha}{\alpha+\beta}, \quad \mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$\alpha = 1$ and $\beta = 1$ gives the Uniform distribution on $(0,1)$.

Gamma distribution with parameters $r > 0$ and $\lambda > 0$:

$$f(x) = \begin{cases} \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases} \qquad E[X] = \frac{r}{\lambda}, \quad \mathrm{Var}(X) = \frac{r}{\lambda^2}$$

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r \quad \text{for } t < \lambda.$$

$r = 1$ gives the Exponential distribution with mean $1/\lambda$.

Normal distribution with mean μ and variance $\sigma^2 > 0$:

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} & \text{if } x \in \mathbb{R} \\ 0 & \text{otherwise.} \end{cases} \qquad E[X] = \mu, \quad \mathrm{Var}(X) = \sigma^2$$

$$M_X(t) = e^{\mu t + t^2\sigma^2/2} \quad \text{for } t \in \mathbb{R}.$$

$\mu = 0$ and $\sigma^2 = 1$ gives the standard normal distribution.

Poisson distribution with mean $\lambda > 0$:

$$f(k) = \begin{cases} \frac{\lambda^k}{k!} e^{-\lambda} & \text{if } k = 0, 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \qquad E[X] = \lambda, \quad \mathrm{Var}(X) = \lambda$$

$$M_X(t) = \exp\{\lambda(e^t - 1)\} \quad \text{for } t \in \mathbb{R}.$$

The distribution function, $\Phi(z)$, of a standard normal random variable

Note: $\Phi(-z) = 1 - \Phi(z)$ for any $z \in \mathbb{R}$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998