Student Number

Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

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- PLEASE NOTE: "Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written."
- "The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer."
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- Formulas and tables are attached. An 8.5×11 inch sheet of notes (both sides) is permitted. The Casio 991 calculator is permitted. HOWEVER, do reasonable simplifications.
- Write the answers in the space provided, continue on the backs of pages if needed.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks. Marks per part question are shown in brackets at the right margin.

Marks: Please do not write in the space below.

Problem 1 [10]	Problem 4 [10]
Problem 2 [10]	Problem 5 [10]
Problem 3 [10]	Problem 6 [10]

Total: [60]

1. Let U and V be independent, discrete random variables, each uniformly distributed on the integers $1, \ldots, n$, i.e., $P(U = i) = P(V = i) = 1/n$, for $i = 1, \ldots, n$. Let $X = U - V$ and $Y = U + V$.

(a) Find $p_X(x)$ and $p_Y(y)$, the marginal pmfs of X and Y, respectively. *Note:* This is an exercise in constraints! [7]

(b) Are X and Y independent? Justify your answer. [3]

2. Let $X_1, \ldots, X_n$ be independent and identically distributed Exponential(λ) random variables. Compute $E[X_{(1)} e^{-\lambda X_{(2)}}]$, where $X_{(1)}$ and $X_{(2)}$ are the first and second order statistics of $X_1, \ldots, X_n$. [10]

3. We have 5 boxes, which are initially empty, and 4 red balls and 4 blue balls. For each ball, we pick a box at random (equally likely) and place the ball in the box, independently from ball to ball. Let

$$\begin{aligned} W &= \text{the number of empty boxes} \\ X &= \text{the number of boxes with no red balls} \\ Y &= \text{the number of boxes with no blue balls} \\ Z &= \text{the number of boxes with at least one red and one blue ball} \end{aligned}$$

(a) Find $\mathrm{Cov}(W, X)$. [7]

(b) Find $\rho(W, X + Y + Z)$, where $\rho(\cdot, \cdot)$ is the correlation coefficient. *Hint:* This is a one (or two) sentence answer. [3]

4. Suppose 2 urns (urn 1 and urn 2) contain N balls in total (so if urn 1 contains i balls then urn 2 contains $N - i$ balls). Balls are drawn independently. On each draw one ball is drawn and is then placed in the other urn (the urn the ball was not drawn from).

(a) Here assume $N = 4$ and a ball is selected at random (i.e., suppose the balls are numbered 1 to 4 and for each draw ball number i is selected with probability $1/4$, for $i = 1, 2, 3, 4$, and then drawn from whichever urn it happens to be in). Let M_i, $i = 0, 1, 2, 3, 4$, denote the expected number of draws until one of the urns is empty, if initially urn 1 contains i balls and urn 2 contains $4 - i$ balls. Find M_i, $i = 1, 2, 3$. Note that $M_0 = M_4 = 0$. [5]

(b) Here assume N is any fixed positive integer but now suppose for each draw we select an *urn* at random (each with probability 1/2) and then select a ball from the chosen urn. Let p_i, $i = 0, \ldots, N$, denote the probability that urn 1 is empty before urn 2 is empty, if initially urn 1 contains i balls and urn 2 contains $N - i$ balls. Find p_i, $i = 1, \ldots, N - 1$. Note that $p_0 = 1$ and $p_N = 0$. [5]

5. Let X and $X_1, X_2, \ldots$ be random variables each having a $N(0,1)$ distribution. Suppose (X_n, X) has a bivariate normal distribution for each n and the correlation between X_n and X is $\rho(X_n, X) = \rho_n$, for $n \geq 1$.

(a) Show that X_n converges to X in distribution as $n \to \infty$ (for arbitrary correlations ρ_n). [2]

(b) If $\rho_n \to 1$ as $n \to \infty$, show that X_n converges to X in probability as $n \to \infty$. [4]

(c) Show that if $\rho_n = 1 - a^n$ for some constant $a \in (0, 1)$, then X_n converges to X with probability 1 as $n \to \infty$. Do you get convergence with probability 1 if $a = 0$? If $a = 1$? Prove your answers. [4]

6. Suppose 80 points are uniformly distributed in the ball in $\mathbb{R}^3$ centred at the origin with radius 1. Let D_i be the Euclidean distance from the origin of the ith point, $i = 1, \ldots, 80$, and let $\overline{D} = \frac{1}{80}\sum_{i=1}^{80} D_i$. Use the central limit theorem to find a value c satisfying $P(|\overline{D} - E[\overline{D}]| \leq c) = .95$. Note that the volume of a ball of radius r is $4\pi r^3/3$. [10]

Formula Sheet

Special Distributions

Exponential distribution with parameter $\lambda > 0$:

$$\begin{aligned} f(x) &= \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases} \qquad E[X] = \frac{1}{\lambda}, \quad \mathrm{Var}(X) = \frac{1}{\lambda^2} \\ F_X(x) &= \begin{cases} 1 - e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Normal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$:

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \\ E[X] &= \mu, \quad \mathrm{Var}(X) = \sigma^2 \end{aligned}$$

The distribution function, $\Phi(z)$, of a standard normal random variable

Note: $\Phi(-z) = 1 - \Phi(z)$ for any $z \in \mathbb{R}$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998