Queen's University
Department of Mathematics and Statistics

MTHE/STAT 353

Final Examination, Part 1 Solutions, 2020
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INSTRUCTIONS:

- The exam is in two parts. This is Part 1 of the exam.
- Part 1 has 3 questions, each worth 10 marks. For each question, begin each solution at the start of a fresh page, and put your student number at the start of each solution.
- The 3 solutions for Part 1 are to be submitted through crowdmark. You should have received an email inviting you to submit your solutions for Part 1 to crowdmark. Upload each solution separately.
- THE DEADLINE FOR SUBMISSION OF PART 1 IS 9:30PM, APRIL 13. **THERE WILL BE ABSOLUTELY NO EXTENSIONS**. YOU MUST ALLOW TIME TO PREPARE YOUR SOLUTIONS FOR UPLOAD TO CROWDMARK, SO PLAN ON COMPLETING THE EXAM BY 9PM (2 HOURS TO WRITE YOUR SOLUTIONS).
- Part 1 of the exam is open book. This means that you can use your notes, the textbook, and your computer. With this said, please note that Part 1 of the exam is designed so that the most effective way to approach this part is to treat it as if it were a proctored, closed book exam.
- **ABSOLUTELY ZERO COLLABORATION IS ALLOWED ON THE EXAM.** There is to be no collaboration in any form on any question on any part of the exam, either in person or remotely. All work on the exam must be completed *on your own*.

Instructions continued on page 2.

- "The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer."
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- You may write your solutions in the space provided, continuing on your own paper if needed, or you may write your solutions using your own paper.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks. Marks per part question are shown in brackets at the right margin.

Problem 1 [10] Problem 2 [10] Problem 3 [10]

Total: [30] _____

Student Number

1. Let X_1, X_2, X_3 be independent Uniform(0,1) random variables, and let $X_{(1)}$, $X_{(2)}$, $X_{(3)}$ denote their order statistics. Let X be the area of the square with side length $X_{(2)}$ and let Y be the area of the rectangle with side lengths $X_{(1)}$ and $X_{(3)}$.

(a) Find $P(X > Y)$ [5]

Solution: We wish to compute $P(X_{(2)}^2 > X_{(1)}X_{(3)})$. The joint pdf of $(X_{(1)}, X_{(2)}, X_{(3)})$ is $f_{123}(x_1, x_2, x_3) = 6$ for $0 < x_1 < x_2 < x_3 < 1$ and equals 0 otherwise. The solution is computed by integrating $f_{123}(x_1, x_2, x_3)$ over the region $A = \{(x_1, x_2, x_2) : x_2^2 - x_1x_3 > 0\}$. If we set the inner integral over x_2 and note that for given values of $0 < x_1 < x_3 < 1$, the possible values of x_2 are $\sqrt{x_1x_3} < x_2 < x_3$. The integral can be written as

$$\begin{aligned}
\iiint\limits_A f_{123}(x_1, x_2, x_3)dx_1dx_2dx_3 &= 6\int_0^1\int_0^{x_3}\int\limits_{\sqrt{x_1x_3}}^{x_3} dx_2dx_1dx_3 \\
&= 6\int_0^1\int_0^{x_3}\left(x_3 - x_1^{1/2}x_3^{1/2}\right)dx_1dx_3 \\
&= 6\int_0^1\left(x_3^2 - x_3^{1/2}\frac{2}{3}x_3^{3/2}\right)dx_3 \\
&= 2\int_0^1 x_3^2dx_3 = \frac{2}{3}.
\end{aligned}$$

(b) Find $E[X]$ and $E[Y]$. [5]

Solution: The marginal pdf of $X_{(2)}$ and the joint pdf of $(X_{(1)}, X_{(3)})$ are given by

$$f_2(x_2) = \begin{cases} 6x_2(1-x_2) & 0 < x_2 < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_{13}(x_1, x_3) = \begin{cases} 6(x_3 - x_1) & 0 < x_1 < x_3 < 1 \\ 0 & \text{otherwise,} \end{cases}$$

respectively. Then

$$\begin{aligned} E[X] &= E[X_{(2)}^2] = \int_0^1 x_2^2 6x_2(1-x_2)dx_2 = \frac{6}{4} - \frac{6}{5} = \frac{3}{10} \\ E[Y] &= E[X_{(1)}X_{(3)}] = \int_0^1 \int_0^{x_3} x_1 x_3 6(x_3 - x_1)dx_1 dx_3 \\ &= 6\int_0^1 \Big(\frac{x_3^4}{2} - \frac{x_3^4}{3}\Big)dx_3 = \frac{1}{5}. \end{aligned}$$

Student Number

2(a). Let X be a continuous random variable with pdf $f(\cdot)$ and cdf $F(\cdot)$. Find $\mathrm{Cov}(F(X), 1 - F(X))$. [4]

Solution: We have $\mathrm{Cov}(F(X), 1 - F(X)) = E[F(X)(1 - F(X))] - E[F(X)]E[1 - F(X)]$. But

$$\begin{aligned} E[F(X)] &= \int_{-\infty}^{\infty} F(x)f(x)dx = \frac{F(x)^2}{2}\bigg|_{-\infty}^{\infty} = \frac{1}{2} \\ E[F(X)^2] &= \int_{-\infty}^{\infty} F(x)^2 f(x)dx = \frac{F(x)^3}{3}\bigg|_{-\infty}^{\infty} = \frac{1}{3}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{Cov}(F(X), 1 - F(X)) &= E[F(X)] - E[F(X)^2] - E[F(X)] + E[F(X)]^2 \\ &= \left(\frac{1}{2}\right)^2 - \frac{1}{3} = -\frac{1}{12}. \end{aligned}$$

(b) Let X and Y be arbitrary random variables, and let $g(\cdot)$ and $h(\cdot)$ be arbitrary real-valued functions defined on $\mathbb{R}$. For each of the following statements say whether it is TRUE or FALSE. If TRUE prove it and if FALSE give a counterexample.

(i) If X and Y are uncorrelated then so are $g(X)$ and $h(Y)$ for any g and h. [3]

(ii) If $g(X)$ and $h(Y)$ are uncorrelated for all g and h then X and Y are uncorrelated. [3]

Solution:

(i) FALSE. Take X to have a $N(0,1)$ distribution. Then $E[X] = 0$ and $E[X^3] = 0$, and take $Y = X^2$. Then $\text{Cov}(X,Y) = \text{Cov}(X,X^2) = E[X^3] - E[X]E[X^2] = 0$. Take $g(X) = X^2$ and $h(Y) = Y$. Then $\text{Cov}(g(X),h(Y)) = \text{Cov}(X^2,X^2) = \text{Var}(X^2) = 2 > 0$, where $\text{Var}(X^2)$ is the variance of a χ^2_1 distribution.

(ii) TRUE. Let A and B be arbitrary subsets of $\mathbb{R}$, and take $g(X) = I_A(X)$ and $h(Y) = I_B(Y)$. Since $\text{Cov}(g(X),h(Y)) = 0$ we have $E[I_A(X)I_B(Y)] = E[I_A(X)]E[I_B(Y)]$, which is equivalent to $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$. Since A and B were arbitrary this implies that X and Y are independent. But then X and Y are uncorrelated.

Student Number

3. A fair die is rolled N times, where N is a random variable. Let X be the total number of times a 2 is rolled.

(a) If N has a Poisson(λ) distribution and N is independent of the outcomes of the rolls, find $P(X = k)$ for $k \geq 0$. (Note that if $N = 0$ then the die is not rolled and $X = 0$). [5]

Solution: Given $N = n$, $X = 0$ if $n = 0$ and X has a Binomial($n, 1/6$) distribution if $n > 0$. Then

$$\begin{aligned} P(X=0) &= \sum_{n=0}^{\infty} P(X=0 \mid N=n)P(N=n) \\ &= e^{-\lambda} + \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n \frac{\lambda^n}{n!} e^{-\lambda} \\ &= e^{-\lambda}(1 + e^{5\lambda/6} - 1) = e^{-\lambda/6}, \end{aligned}$$

and for $k > 0$

$$\begin{aligned} P(X=k) &= \sum_{n=k}^{\infty} P(X=k \mid N=n)P(N=n) \\ &= \sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k} \frac{\lambda^n}{n!} e^{-\lambda} \\ &= \frac{1}{k!} \left(\frac{1}{6}\right)^k \lambda^k e^{-\lambda} \sum_{n=k}^{\infty} \frac{(5\lambda/6)^{n-k}}{(n-k)!} \\ &= \frac{(\lambda/6)^k}{k!} e^{-\lambda} e^{5\lambda/6} \\ &= \frac{(\lambda/6)^k}{k!} e^{-\lambda/6}. \end{aligned}$$

(So $X \sim$ Poisson($\lambda/6$)).

(b) If N is the first roll on which a one is rolled, find $E[X]$ and $\mathrm{Var}(X)$. [5]

Solution: The distribution of N is Geometric(1/6) on the positive integers. Also, given $N = 1$ we must have $X = 0$, and, for $n > 1$, the conditional distribution of X given $N = n$ is Binomial$(n-1, 1/5)$. To see this, note that the event $\{N = n\}$ is the event $\{X_1 \neq 1, \ldots, X_{n-1} \neq 1, X_n = 1\}$, where X_i denotes the outcome of the ith roll. Given this, each X_i is discrete uniform on $\{2, \ldots, 5\}$ for $i = 1, \ldots, n-1$ and $X_n = 1$ (easy to check) and $X_1, \ldots, X_{n-1}$ are independent. Then $X = \sum_{i=1}^{N} I_{\{2\}}(X_i)$ given $N = n$ will be Binomial$(n-1, 1/5)$. So $E[X \mid N = n] = \frac{1}{5}(n-1)$ (also holds for $n = 1$) and $\mathrm{Var}(X \mid N = n) = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)(n-1)$ (which also holds for $n = 1$). Then with $E[N] = 1/(1/6) = 6$ and by the law of total expectation

$$E[X] = E[E[X \mid N]] = E\Big[\frac{1}{5}(N-1)\Big] = \frac{1}{5}(E[N]-1) = \frac{1}{5}(6-1) = 1$$

and with $\mathrm{Var}(N) = (1-1/6)/(1/6)^2 = 30$ and by the conditional variance formula

$$\begin{aligned}
\mathrm{Var}(X) &= E[\mathrm{Var}(X \mid N)] + \mathrm{Var}(E[X \mid N]) \\
&= \frac{4}{25}(E[N]-1) + \frac{1}{25}\mathrm{Var}(N) \\
&= \frac{4}{5} + \frac{6}{5} = 2.
\end{aligned}$$