Student Number

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Department of Mathematics and Statistics

STAT 353

Final Examination April 9, 2009
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- "Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written."
- "The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer."
- Formulas and tables are attached.
- An 8.5×11 inch sheet of notes (both sides) is permitted.
- Simple calculators are permitted. HOWEVER, do reasonable simplifications.
- Write the answers in the space provided, continue on the backs of pages if needed.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.
- Marks per part question are shown in brackets at the right margin.

Marks: Please do not write in the space below.

Problem 1 [10]	Problem 4 [10]
Problem 2 [10]	Problem 5 [10]
Problem 3 [10]	Problem 6 [10]

Total: [60]

1. Let X and Y be independent random variables, where X has a Poisson distribution with parameter 1 and Y has an exponential distribution with parameter 1. Show that

$$E\left[\left(\frac{Y}{2}\right)^X\right] = \frac{2}{e}.$$

Hint: You can condition on X and compute $E[Y^n]$ for any nonnegative integer n. [10]

2. (a) Let X and Y be independent random variables each having the uniform distribution on $(0, 1)$. Let $U = \min(X, Y)$ and $V = \max(X, Y)$. Compute $\mathrm{Cov}(U, V)$. *Hint:* Note that $UV = XY$ with probability 1. [6]

(b) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with finite variance, and let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Show that $\mathrm{Cov}(\overline{X}_n, X_i - \overline{X}_n) = 0$ for all $i = 1, \ldots, n$. [4]

3. Let Y be a continuous random variable with probability density function $f_Y(y)$ and moment generating function $M_Y(t)$ (assume $|M_Y(t)| < \infty$ for all t). For a fixed τ, let X have probability density function

$$f_X(x) = \frac{e^{\tau x} f_Y(x)}{M_Y(\tau)}.$$

(a) Find $M_X(t)$, the moment generating function of X. [4]

(b) Suppose that $Y \sim N(\mu, \sigma^2)$. Find $E[X]$. [6]

4. Consider a sequence of independent experiments, where in each experiment we take k balls, labelled 1 to k and randomly place them into k slots, also labelled 1 to k, so that there is exactly one ball in each slot. For the ith experiment, let X_i be the number of balls whose label matches the slot label of the slot into which it is placed. So $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables.

(a) Find the mean and variance of X_i. *Hint:* Write X_i as $X_i = X_{i1} + \ldots + X_{ik}$, where X_{ij} is the indicator that slot j receives ball j in the ith experiment. [6]

(b) Use the central limit theorem to approximate the probability that in the first 25 experiments the total number of balls whose label matches their slot label is greater than 30. [4]

5. Let $Y_1, Y_2, \ldots$ be a sequence of discrete random variables such that the joint probability mass function of $(Y_1, \ldots, Y_n)$ is

$$P(Y_1 = y_1, \ldots, Y_n = y_n) = \begin{cases} \left[(n+1)\binom{n}{\sum_{i=1}^n y_i}\right]^{-1} & \text{for } y_i \in \{0,1\},\ i = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

(so the Y_i's are Bernoulli random variables but they are not independent). Let X_n be the sample mean of $Y_1, \ldots, Y_n$, i.e., $X_n = \frac{1}{n}\sum_{i=1}^n Y_i$. Show that X_n converges in distribution to a limit X and find the distribution of X. *Hint:* X is not a constant. *Hint:* First find the distribution of $\sum_{i=1}^n Y_i$, which has sample space $\{0, 1, \ldots, n\}$. [10]

6. Let $\{Y(t) : t \geq 0\}$ be a Brownian motion with drift parameter μ and variance parameter σ^2, and define $X(t) = e^{Y(t)}$ (i.e., $\{X(t) : t \geq 0\}$ is a Geometric Brownian motion). For $0 < s < t$, show that

$$E[X(t) \mid X(u), 0 \leq u \leq s] = X(s)e^{(t-s)(\mu+\sigma^2/2)}$$

[10]

Formula Sheet

Special Distributions

Continuous uniform on (a, b):

$$f(x) = \begin{cases} 1/(b-a) & \text{if } a \le x \le b \\ 0 & \text{otherwise,} \end{cases} \qquad E[X] = \frac{a+b}{2}, \quad \mathrm{Var}[X] = \frac{(b-a)^2}{12}.$$

Exponential with parameter λ:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases} \qquad E[X] = \frac{1}{\lambda}, \quad \mathrm{Var}[X] = \frac{1}{\lambda^2}.$$

Gamma with parameters r and λ:

$$f(x) = \begin{cases} \frac{\lambda^r}{\Gamma(r)} x^{\alpha-1} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases} \qquad E[X] = \frac{r}{\lambda}, \quad \mathrm{Var}[X] = \frac{r}{\lambda^2}.$$

Normal (Gaussian) with mean μ and variance σ^2:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{and} \quad F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\, dt.$$

Poisson with parameter λ:

$$P(X = k) = \begin{cases} \frac{\lambda^k}{k!} e^{-\lambda} & k = 0, 1, \ldots \\ 0 & \text{otherwise.} \end{cases} \qquad E[X] = \lambda, \quad \mathrm{Var}[X] = \lambda.$$

- df and pdf of the kth order statistic from a random sample $X_1, \ldots, X_n$:

$$\begin{aligned} F_k(x) &= \sum_{i=k}^{n} \binom{n}{i} [F(x)]^i [1 - F(x)]^{n-i}; \\ f_k(x) &= \frac{n!}{(k-1)!(n-k)!} f(x) [F(x)]^{k-1} [1 - F(x)]^{n-k}, \end{aligned}$$

where $F(x)$ and $f(x)$ are the df and pdf, respectively, of each X_i.

The distribution function of a standard normal random variable