Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Homework 1 Due Thursday, January 27, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  (1) Start your solution in the space provided right after the problem statement, and use your own paper if you need extra pages.
  (2) Write your whole solution using your own paper, and make sure to number your solution.
  (3) Write your solution using document creation software (e.g., Word or LaTeX).
- Write your name and student number on the first page of each solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 25

Student Number Name

1. (5 marks) Suppose an urn initially contains one red ball, one blue ball, and one green ball. At each draw, a ball is randomly selected from the urn, replaced, and an additional ball of the same colour as the drawn ball is added to the urn. Thus, after n draws there are $n+3$ balls in the urn. After n draws, let X be the number of times a red ball was drawn, Y the number of times a blue ball was drawn, and Z the number of times a green ball was drawn. Compute the joint probability mass function of the random vector (X, Y, Z). As part of the joint pmf you must also give the support of the distribution!

Student Number Name

2. (5 marks) Let X_1, X_2, X_3 be discrete random variables with joint pmf

$$p_X(x_1, x_2, x_3) = \left(\frac{1}{2}\right)^{x_3} (1 - e^{-x_3})^2 e^{-x_3(x_1+x_2-2)},$$

for $x_1, x_2, x_3 = 1, 2, 3, \ldots$ and $p_X(x_1, x_2, x_3) = 0$ otherwise. Find the marginal pmf of X_1.

Student Number Name

3. (6 marks) Let X_1, X_2, X_3 be continuous random variables with joint pdf

$$f_X(x_1, x_2, x_3) = \frac{1}{\sqrt{2\pi}} e^{-(x_1-x_3)^2/2} \frac{1}{\sqrt{2\pi}} e^{-(x_2-x_3)^2/2} \frac{1}{\sqrt{2\pi}} e^{-x_3^2/2},$$

for $-\infty < x_1, x_2, x_3 < \infty$. Find the joint marginal pdf of (X_1, X_2) and the marginal pdf of X_1.

Student Number   Name

4. (4 marks) An urn contains 2 balls numbered '1', 2 balls numbered '2', and 10 balls numbered '3'. Seven balls are drawn at random from the urn, without replacement. Let X_i be the number of balls in the sample that are numbered 'i', for $i = 1, 2, 3$. Find $E[X_3]$.

Student Number

Name

5. (5 marks) We have seen in the class lecture notes that if $X_1, \ldots, X_n$ are independent then $g_1(X_1), \ldots, g_n(X_n)$ are also independent, where $g_1(\cdot), \ldots, g_n(\cdot)$ are arbitrary real-valued functions. Hence, it follows that $E[g_1(X_1) \ldots g_n(X_n)] = E[g_1(X_1)] \ldots E[g_n(X_n)]$ (assuming the expectations exist). Conversely, show that if

$$E[g_1(X_1) \ldots g_n(X_n)] = E[g_1(X_1)] \ldots E[g_n(X_n)]$$

holds for all functions $g_1, \ldots, g_n$ for which the expectations exist, then $X_1, \ldots, X_n$ are mutually independent. *Hint:* Consider functions which are indicators of sets.