Queen’s University
Department of Mathematics and Statistics

MTHE/STAT 353
Homework 1 Solutions, 2020

• For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:

   (1) Start your solution in the space provided right after the problem statement, and use your own paper if you need extra pages.

   (2) Write your whole solution using your own paper, and make sure to number your solution.

   (3) Write your solution using document creation software (e.g., Word or LaTeX).

• Write your name and student number on the first page of each solution.

• For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Total Marks: 26
1. Let $X_1, X_2, X_3$ be discrete random variables with joint pmf

$$p_X(x_1, x_2, x_3) = \left(\frac{1}{2}\right)^{x_3} (1 - e^{-x_3})^2 e^{-x_3(x_1 + x_2 - 2)},$$

for $x_1, x_2, x_3 = 1, 2, 3, \ldots$ and $p_X(x_1, x_2, x_3) = 0$ otherwise. Find the marginal pmf of $X_1$

**Solution:** (5 marks) Let $x_1 \in \mathbb{N}$ be fixed. Sum over $x_2$ first:

$$\sum_{x_2=1}^{\infty} p_X(x_1, x_2, x_3) = \left(\frac{1}{2}\right)^{x_3} (1 - e^{-x_3})e^{-x_3(x_1 - 1)} \sum_{x_2=1}^{\infty} (1 - e^{-x_3})^{x_2 - 1}$$

$$= \left(\frac{1}{2}\right)^{x_3} (1 - e^{-x_3})e^{-x_3(x_1 - 1)},$$

since the sum, being the sum over all probabilities of a Geometric$(1 - e^{-x_3})$ distribution, is equal to one. Next, sum over $x_3$:

$$\sum_{x_3=1}^{\infty} \sum_{x_2=1}^{\infty} p_X(x_1, x_2, x_3) = \sum_{x_3=1}^{\infty} \left(\frac{1}{2}\right)^{x_3} (1 - e^{-x_3})e^{-x_3(x_1 - 1)}$$

$$= \sum_{x_3=1}^{\infty} \left(\frac{e^{-(x_1 - 1)}}{2}\right)^{x_3} - \sum_{x_3=1}^{\infty} \left(\frac{e^{-x_1}}{2}\right)^{x_3}$$

$$= \frac{1}{1 - e^{-(x_1 - 1)}} - 1 - \left(\frac{1}{1 - e^{-x_1}} - 1\right)$$

$$= \frac{2}{2 - e^{-(x_1 - 1)}} - \frac{2}{2 - e^{-x_1}}.$$

So the marginal pmf of $X_1$ is

$$p_{X_1}(x_1) = \left\{ \begin{array}{ll}
\frac{2}{2 - e^{-(x_1 - 1)}} - \frac{2}{2 - e^{-x_1}} & \text{for } x_1 = 1, 2, 3, \ldots \\
0 & \text{otherwise.}
\end{array} \right.$$
2. Let \( X_1, X_2, X_3 \) be continuous random variables with joint pdf

\[
f_X(x_1, x_2, x_3) = \frac{1}{\sqrt{2\pi}} e^{-(x_1-x_3)^2/2} \frac{1}{\sqrt{2\pi}} e^{-(x_2-x_3)^2/2} \frac{1}{\sqrt{2\pi}} e^{-x_3^2/2},
\]

for \(-\infty < x_1, x_2, x_3 < \infty\). Find the joint marginal pdf of \((X_1, X_2)\) and the marginal pdf of \(X_1\).

Solution: (6 marks) To get the joint marginal pdf of \((X_1, X_3)\) we need to integrate out \(x_2\). Combining the exponents above we get a quadratic function of \(x_3\), which we will write as \(-\frac{1}{2a}(x_3 - b)^2\), where \(a\) and \(b\) do not depend on \(x_3\):

\[
-\frac{(x_1 - x_3)^2}{2} - \frac{(x_2 - x_3)^2}{2} - \frac{x_3^2}{2} = -\frac{1}{2}(x_1^2 - 2x_1x_3 + x_3^2 + x_2^2 - 2x_2x_3 + x_3^2 + x_3^2) = -\frac{1}{2}(3x_3^2 - 2x_3(x_1 + x_2) + x_1^2 + x_2^2) = -\frac{3}{2} \left( x_3^2 - 2x_3 \frac{x_1 + x_2}{3} + \frac{x_1^2 + x_2^2}{3} \right) = -\frac{3}{2} \left( x_3 - \frac{x_1 + x_2}{3} \right)^2 - \frac{3}{2} \left( \frac{x_1^2 + x_2^2}{3} - \frac{(x_1 + x_2)^2}{9} \right) = -\frac{1}{2a} (x_3 - b)^2 - \frac{3}{2} \left( \frac{x_1^2 + x_2^2}{3} - \frac{(x_1 + x_2)^2}{9} \right),
\]

where \(a = 1/3\) and \(b = \frac{x_1 + x_2}{3}\). This shows that for fixed \(x_1\) and \(x_2\), the joint pdf \(f_X(x_1, x_2, x_3)\) as a function of \(x_3\) is a constant, say \(c\), times a \(N(b, a)\) density for \(x_3\). Integrating over all \(x_3\) leaves just the constant \(c\), which depends on \(x_1\) and \(x_2\). Then \(c\) considered as a function of \(x_1\) and \(x_2\) is the joint marginal pdf of \((X_1, X_2)\), which is

\[
f_{X_1, X_2}(x_1, x_2) = \frac{\sqrt{a}}{2\pi} \exp \left\{ -\frac{3}{2} \left( \frac{x_1^2 + x_2^2}{3} - \frac{(x_1 + x_2)^2}{9} \right) \right\} = \frac{1}{2\pi\sqrt{3}} \exp \left\{ -\frac{1}{6} \left( 3x_1^2 + 3x_2^2 - x_1^2 - 2x_1x_2 - x_2^2 \right) \right\} = \frac{1}{2\pi\sqrt{3}} \exp \left\{ -\frac{1}{3} \left( x_1^2 - x_1x_2 + x_2^2 \right) \right\},
\]

which is valid for \(-\infty < x_1, x_2 < \infty\). For the marginal pdf of \(X_1\) we integrate \(f_{X_1, X_2}(x_1, x_2)\) over all \(x_2\). We write the exponent as

\[
-\frac{1}{3} (x_1^2 - x_1x_2 + x_2^2) = -\frac{1}{3} \left( x_2^2 - 2\frac{x_1x_2}{2} + x_1^2 \right) = -\frac{1}{3} \left( \left( x_2 - \frac{x_1}{2} \right)^2 + x_1^2 - \frac{x_1^2}{4} \right)
\]

So, for fixed \(x_1\), \(f_{X_1, X_2}(x_1, x_2)\) is a constant times a \(N\left(\frac{x_1}{2}, \frac{3}{2}\right)\) density for \(x_2\). Integrating out \(x_2\) leaves the constant, which as a function of \(x_1\) gives the marginal pdf of \(X_1\):

\[
f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi\sqrt{3}}} \sqrt{\frac{3}{2}} \exp \left\{ -\frac{1}{3} \left( x_1^2 - \frac{x_1^2}{4} \right) \right\} = \frac{1}{\sqrt{2\pi\sqrt{2}}} \exp \left\{ -\frac{1}{2(2)} x_1^2 \right\},
\]

i.e., \(X_1\) has a \(N(0, 2)\) distribution.
3. An urn contains 2 balls numbered '1', 2 balls numbered '2', and 10 balls numbered '3'. Seven balls are drawn at random from the urn, without replacement. Let $X_i$ be the number of balls in the sample that are numbered '$i$', for $i = 1, 2, 3$. Find $E[X_3]$.

**Solution: (4 marks)** We first get the marginal pmf of $X_3$. The possible values of $X_3$ are 3, 4, 5, 6 and 7. Relabel the balls numbered '1' and '2' as '0', so the urn contains 4 balls labelled '0' and 10 balls labelled '3'. Let $X_0$ denote the number of balls in the sample that are labelled '0'. For $x_3 \in \{3, 4, 5, 6, 7\}$,

$$P(X_3 = x_3) = P(X_3 = x_3, X_0 = 7 - x_3) = \binom{10}{x_3} \binom{4}{7 - x_3} \binom{14}{7}.$$

Then

$$E[X_3] = \sum_{x_3=3}^{7} x_3 \frac{\binom{10}{x_3} \binom{4}{7-x_3}}{\binom{14}{7}}.$$

$$= \frac{1}{\binom{14}{7}} \left[ 3\binom{10}{3} \binom{4}{4} + 4\binom{10}{4} \binom{4}{3} + 5\binom{10}{5} \binom{4}{2} + 6\binom{10}{6} \binom{4}{1} + 7\binom{10}{7} \binom{4}{0} \right]$$

$$= \frac{1}{3432} \left( 3(120)(1) + 4(210)(4) + 5(252)(6) + 6(210)(4) + 7(120)(1) \right)$$

$$= \frac{17160}{3432} = 5.$$
4. Let $X_1, X_2, X_3$ be independent, discrete random variables, where $X_1$ and $X_2$ are both distributed as Poisson($\theta$) and $X_3$ has a Geometric(1/2) distribution (on the positive integers, so $X_3$ has pmf $f_3(x_3) = (1/2)^x_3$ for $x_3 = 1, 2, \ldots$ and $f_3(x_3) = 0$ otherwise). Find the probability that the random matrix
\[
\begin{bmatrix}
X_1 & 0 & 0 \\
0 & X_2 & X_2 \\
0 & X_3 & X_2
\end{bmatrix}
\]
is singular.

Solution: (5 marks) Let $A$ be the event that the matrix is singular (which is the same as the event that the determinant of the matrix is 0). Then $P(A) = P(A_1 \cup A_2 \cup A_3)$, where $A_1 = \{X_1 = 0\}$, $A_2 = \{X_2 = 0\}$, and $A_3 = \{X_2 = X_3 \neq 0\}$. The event $A_3$ is disjoint from $A_2$, and so
\[
P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3).
\]
From the given distributions for $X_1$, $X_2$, and $X_3$ and by their mutual independence, we have
\[
P(A_1) = e^{-\theta}
\]
\[
P(A_2) = e^{-\theta}
\]
\[
P(A_1 \cap A_2) = P(A_1)P(A_2) = e^{-2\theta}
\]
\[
P(A_3) = \sum_{k=1}^{\infty} P(X_2 = k, X_3 = k) = \sum_{k=1}^{\infty} \frac{\theta^k}{k!} e^{-\theta} \left(\frac{1}{2}\right)^k = e^{-\theta}(e^{\theta/2} - 1) = e^{-\theta/2} - e^{-\theta}
\]
\[
P(A_1 \cap A_3) = P(A_1)P(A_3) = e^{-3\theta/2} - e^{-2\theta}
\]
So we get
\[
P(A) = e^{-\theta} + e^{-\theta} - e^{-2\theta} + e^{-\theta/2} - e^{-\theta} - (e^{-3\theta/2} - e^{-2\theta})
\]
\[
= e^{-\theta} + e^{-\theta/2} - e^{-3\theta/2}.
\]
5. Suppose $X_1, \ldots, X_n$ are independent random variables, and $X_k \sim \text{Exponential}(k)$, i.e., $X_k$ has pdf

$$f_k(x_k) = \begin{cases} ke^{-kx_k} & \text{for } x_k \geq 0 \\ 0 & \text{for } x_k < 0 \end{cases},$$

for $k = 1, \ldots, n$. Find $P(\min(X_1, \ldots, X_n) = X_n)$ and $P(X_n < X_{n-1} < \ldots < X_2 < X_1)$.

**Solution:** (6 marks) Let $A$ denote the event “$\min(X_1, \ldots, X_n) = X_n$”. Then $A = \{X_n < X_1, \ldots, X_n < X_{n-1}\}$. The $n$-dimensional integral giving $P(A)$ is

$$P(A) = \int_0^\infty \int_0^{\infty} \ldots \int_0^{\infty} (e^{-x_1})(2e^{-2x_2}) \ldots (ne^{-nx_n}) dx_1 \ldots d_{x_{n-1}} dx_n$$

$$= \int_0^\infty \int_0^{\infty} \ldots \int_0^{\infty} (2e^{-2x_2}) \ldots (ne^{-nx_n}) e^{-x_2} dx_2 \ldots d_{x_{n-1}} dx_n$$

$$= \int_0^\infty \int_0^{\infty} \ldots \int_0^{\infty} (3e^{-3x_3}) \ldots (ne^{-nx_n}) e^{-x_3} e^{-2x_2} dx_3 \ldots d_{x_{n-1}} dx_n$$

$$\vdots$$

$$= \int_0^\infty ne^{-nx_n} e^{-x_n} e^{-2x_n} \ldots e^{-(n-1)x_n} dx_n$$

$$= n \int_0^\infty e^{-(n(n+1)/2)x_n} dx_n = \frac{2n}{n(n+1)} = \frac{2}{n+1}.$$

Let $B$ denote the event $\{X_n < X_{n-1} < \ldots < X_1\}$. Then

$$P(B) = \int_0^\infty \int_0^{\infty} \ldots \int_0^{\infty} (e^{-x_1})(2e^{-2x_2}) \ldots (ne^{-nx_n}) dx_1 \ldots d_{x_{n-1}} dx_n$$

$$= n! \int_0^\infty \int_0^{\infty} \ldots \int_0^{\infty} (e^{-x_2})(e^{-2x_3}) \ldots (e^{-nx_n}) dx_2 \ldots d_{x_{n-1}} dx_n$$

$$= n! \frac{2}{3} \int_0^\infty \int_0^{\infty} \ldots \int_0^{\infty} (e^{-3x_3})(e^{-3x_4}) \ldots (e^{-nx_n}) dx_3 \ldots d_{x_{n-1}} dx_n$$

$$\vdots$$

$$= \frac{n!}{(3)(6)(10) \ldots (1 + 2 + \ldots + n - 1)} \int_0^\infty e^{-(1+2+\ldots+n-1)x_n} e^{-nx_n} dx_n$$

$$= \frac{n!}{(3)(6)(10) \ldots (1 + 2 + \ldots + n - 1) n(n+1)}.$$

Now,

$$(3)(6)(10) \ldots (1 + 2 + \ldots n - 1) = \prod_{k=2}^{n-1} \frac{k(k + 1)}{2} = \frac{(n - 1)!n!}{2^{n-1}},$$

so we get

$$P(B) = \frac{2^n}{(n+1)!}.$$