Queen's University
Department of Mathematics and Statistics

MTHE/STAT 353

Homework 2 Due Thursday, February 3, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  (1) Start your solution in the space provided right after the problem statement, and use your own paper if you need extra pages.
  (2) Write your whole solution using your own paper, and make sure to number your solution.
  (3) Write your solution using document creation software (e.g., Word or LaTeX).
- Write your name and student number on the first page of each solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 28

Student Number ________________ Name ________________

1. (6 marks) Suppose that the random vector $X = (X_1, X_2, X_3)^T$ is uniformly distributed on the sphere of radius r centred at the origin; that is, X has joint probability density function

$$f_X(x_1, x_2, x_3) = \begin{cases} \frac{3}{4\pi r^3} & \text{if } (x_1, x_2, x_3) \in S \\ 0 & \text{otherwise,} \end{cases}$$

where $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq r^2\}$ is the sphere of radius r centred at $(0, 0, 0)$. Let $Y = X_1^2 + X_2^2 + X_3^2$. Find the distribution function of Y, the probability density function of Y, and $E[Y]$.

Student Number Name

2. (6 marks) Consider a square with edges labelled 1, 2, 3, 4. Each vertex of the square has a value which is randomly chosen from the set $\{1, 2, 3, 4, 5, 6\}$, independently from vertex to vertex. Let

$$X_i = \begin{cases} 1 & \text{if edge } i \text{ connects two vertices with the same value} \\ 0 & \text{otherwise.} \end{cases}$$

Show that X_1, X_2, X_3, X_4 are pairwise independent but are not mutually independent.

Student Number   Name

3. (5 marks) A *product set* in $\mathbb{R}^n$ is a set S of the form $S = S_1 \times S_2 \times \ldots \times S_n$, where $S_i \subset \mathbb{R}$ for $i = 1, \ldots, n$, and the set $S_1 \times \ldots \times S_n$ is defined as

$$S_1 \times \ldots \times S_n = \{(x_1, \ldots x_n) \in \mathbb{R}^n : x_i \in S_i \text{ for } i = 1, \ldots, n\}.$$

Suppose X_i is a discrete random variable with support S_i (i.e., $P(X_i = x) > 0$ if and only if $x \in S_i$), for $i = 1, \ldots, n$, and suppose that the random vector $(X_1, \ldots, X_n)$ has support S with $S \neq S_1 \times \ldots \times S_n$. Show that $X_1, \ldots, X_n$ cannot be mutually independent.

Student Number Name

4. (6 marks) Suppose that $X_1, \ldots, X_n$ are independent and identically distributed continuous random variables, each with probability density function

$$f(x) = \begin{cases} 3/(x+1)^4 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Find $E[X_1^2 X_2^2 \ldots X_n^2]$.

Student Number Name

5. (5 marks) Suppose $X_1, \ldots, X_n$ are independent random variables, and $X_k \sim \text{Exponential}(k)$, i.e., X_k has pdf

$$f_k(x_k) = \begin{cases} ke^{-kx_k} & \text{for } x_k \geq 0 \\ 0 & \text{for } x_k < 0 \end{cases},$$

for $k = 1, \ldots, n$. Find $P(\min(X_1, \ldots, X_n) = X_n)$.