Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Homework 2 Solutions, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  (1) Start your solution in the space provided right after the problem statement, and use your own paper if you need extra pages.
  (2) Write your whole solution using your own paper, and make sure to number your solution.
  (3) Write your solution using document creation software (e.g., Word or LaTeX).
- Write your name and student number on the first page of each solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 28

Student Number Name

1. (6 marks) Suppose that the random vector $X = (X_1, X_2, X_3)^T$ is uniformly distributed on the sphere of radius r centred at the origin; that is, X has joint probability density function

$$f_X(x_1, x_2, x_3) = \begin{cases} \frac{3}{4\pi r^3} & \text{if } (x_1, x_2, x_3) \in S \\ 0 & \text{otherwise,} \end{cases}$$

where $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq r^2\}$ is the sphere of radius r centred at $(0, 0, 0)$. Let $Y = X_1^2 + X_2^2 + X_3^2$. Find the distribution function of Y, the probability density function of Y, and $E[Y]$.

Solution: We first find the df of Y. Let F_Y denote the df of Y and let f_Y denote the pdf of Y. For $y \in [0, r^2]$ we have

$$F_Y(y) = P(Y \leq y) = P\left(\sqrt{X_1^2 + X_2^2 + X_3^2} \leq \sqrt{y}\right) = P((X_1, X_2, X_3) \in S_{\sqrt{y}}),$$

where $S_{\sqrt{y}}$ is the sphere of radius $\sqrt{y}$. Thus, for $y \in [0, r^2]$

$$\begin{aligned} F_Y(y) &= \iiint\limits_{S_{\sqrt{y}}} f_X(x_1, x_2, x_3) dx_1 dx_2 dx_3 = \iiint\limits_{S_{\sqrt{y}}} \frac{3}{4\pi r^3} dx_1 dx_2 dx_3 \\ &= \frac{3}{4\pi r^3} \times \text{Volume of } S_{\sqrt{y}} \\ &= \frac{3}{4\pi r^3} \times \frac{4\pi(\sqrt{y})^3}{3} = \frac{y^{3/2}}{r^3}. \end{aligned}$$

We also have $F_Y(y) = 0$ for $y < 0$ and $F_Y(y) = 1$ for $y > r^2$. Differentiating, we obtain the pdf of Y as

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{3\sqrt{y}}{2r^3} & \text{for } 0 \leq y \leq r^2 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$E[Y] = \frac{3}{2r^3} \int_0^{r^2} y^{3/2} dy = \frac{3}{2r^3} \times \frac{2}{5} y^{5/2} \Big|_0^{r^2} = \frac{3r^5}{5r^3} = \frac{3r^2}{5}.$$

Student Number Name

2. (6 marks) Consider a square with edges labelled 1, 2, 3, 4. Each vertex of the square has a value which is randomly chosen from the set $\{1, 2, 3, 4, 5, 6\}$, independently from vertex to vertex. Let

$$X_i = \begin{cases} 1 & \text{if edge } i \text{ connects two vertices with the same value} \\ 0 & \text{otherwise.} \end{cases}$$

Show that X_1, X_2, X_3, X_4 are pairwise independent but are not mutually independent.

Solution: First, the marginal distribution of each X_i is given by

$$\begin{aligned} P(X_i = 1) &= P(\text{both endpoints of edge } i \text{ have the same value}) \\ &= \sum_{k=1}^{6} P(\text{both endpoints of edge } i \text{ have value } k) = \sum_{k=1}^{6} \left(\frac{1}{6}\right)^2 = \frac{6}{36} = \frac{1}{6}, \end{aligned}$$

and $P(X_i = 0) = 1 - 1/6 = 5/6$. Let us now consider pairs X_i and X_j. First, if i and j are opposite edges of the square then X_i and X_j are independent because they are functions of the vertex values of disjoint sets of vertices, and all the vertex values are independent. So suppose i and j are adjacent edges, that is they share a common vertex. Let V_1 and V_2 denote the vertex values of the endpoints of edge i and V_2 and V_3 denote the vertex values of the endpoints of edge j (so V_2 is the common vertex value). We have

$$\begin{aligned} P(X_i = 1, X_j = 1) &= P(\text{all three vertices } V_1,\ V_2, \text{ and } V_3 \text{ have the same value}) \\ &= \sum_{k=1}^{6} P(\text{all three vertices have the value } k) = \sum_{k=1}^{6} \left(\frac{1}{6}\right)^3 = \frac{1}{36}. \end{aligned}$$

Therefore, $P(X_i = 1, X_j = 1) = 1/36 = P(X_i = 1)P(X_j = 1)$. Note that this is sufficient to show that X_i and X_j are independent because all other joint probabilities involving X_i and X_j are determined by $P(X_i = 1, X_j = 1)$ and $P(X_j = 1)$. For example,

$$P(X_i = 0, X_j = 1) = P(X_j = 1) - P(X_i = 1, X_j = 1) = \frac{1}{6} - \frac{1}{36} = \frac{5}{36},$$

which is the same as $P(X_i = 0)P(X_j = 1)$. The other joint probabilities, $P(X_i = 1, X_j = 0)$ and $P(X_i = 0, X_j = 0)$ can be checked similarly. Finally, we show that X_1, X_2, X_3, X_4 are not mutually independent. For example,

$$P(X_i = 1 \text{ for all } i) = P(\text{all vertices have the same value}) = \sum_{k=1}^{6} \left(\frac{1}{6}\right)^4 = \frac{1}{216}.$$

On the other hand,

$$P(X_1 = 1)P(X_2 = 1)P(X_3 = 1)P(X_4 = 1) = \left(\frac{1}{6}\right)^4 = \frac{1}{1296}.$$

Student Number Name

3. (5 marks) A *product set* in $\mathbb{R}^n$ is a set S of the form $S = S_1 \times S_2 \times \ldots \times S_n$, where $S_i \subset \mathbb{R}$ for $i = 1, \ldots, n$, and the set $S_1 \times \ldots \times S_n$ is defined as

$$S_1 \times \ldots \times S_n = \{(x_1, \ldots x_n) \in \mathbb{R}^n : x_i \in S_i \text{ for } i = 1, \ldots, n\}.$$

Suppose X_i is a discrete random variable with support S_i (i.e., $P(X_i = x) > 0$ if and only if $x \in S_i$), for $i = 1, \ldots, n$, and suppose that the random vector $(X_1, \ldots, X_n)$ has support S with $S \neq S_1 \times \ldots \times S_n$. Show that $X_1, \ldots, X_n$ cannot be mutually independent.

Solution: First, we note that $S \subseteq S_1 \times \ldots \times S_n$. For if it is not, there must be some point $(x_1, \ldots, x_n) \in S$ such that $x_i \notin S_i$ for some i. But since for such an i, $P(X_i = x_i) \geq P(X_1 = x_1, \ldots, X_n = x_n) > 0$, we have that x_i must be in the sample space S_i of X_i, a contradiction. Thus, if $S \neq S_1 \times \ldots \times S_n$ then S must be properly contained inside $S_1 \times \ldots \times S_n$. Hence, there must be at least one point $(y_1, \ldots, y_n) \in S_1 \times \ldots \times S_n$ such that $(y_1, \ldots, y_n) \notin S$. For such a point, we have

$$P(X_1 = y_1, \ldots, X_n = y_n) = 0 \neq P(X_1 = y_1) \times \ldots \times P(X_n = y_n),$$

as the product of marginal probabilities above is positive, since y_i is in the sample space of X_i for each i. Hence, $X_1, \ldots, X_n$ cannot be independent.

Student Number Name

4. (6 marks) Suppose that $X_1, \ldots, X_n$ are independent and identically distributed continuous random variables, each with probability density function

$$f(x) = \begin{cases} 3/(x+1)^4 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Find $E[X_1^2 X_2^2 \ldots X_n^2]$.

Solution: First, we compute $E[X_1^2]$ as follows:

$$\begin{aligned} E[X_1^2] &= \int_0^\infty \frac{3x^2}{(x+1)^4} dx \\ &= 3\int_0^\infty \left[\frac{(x+1)^2 - 2(x+1) + 1}{(x+1)^4}\right] dx \\ &= 3\int_0^\infty \frac{1}{(x+1)^2} dx - 6\int_0^\infty \frac{1}{(x+1)^3} dx + 3\int_0^\infty \frac{1}{(x+1)^4} dx \\ &= 3\left[\frac{-1}{x+1}\right]_0^\infty - 6\left[\frac{-1}{2(x+1)^2}\right]_0^\infty + 3\left[\frac{-1}{3(x+1)^3}\right]_0^\infty \\ &= 3 - 3 + 1 = 1. \end{aligned}$$

Since $X_1, \ldots, X_n$ are mutually independent, so are $X_1^2, \ldots, X_n^2$, and so we have that $E[X_1^2 \ldots X_n^2] = E[X_1^2] \times \ldots \times E[X_n^2]$. Also, since the X_i are identically distributed we have that $E[X_i^2] = E[X_1^2] = 1$ for all i. Therefore,

$$E[X_1^2 \ldots X_n^2] = 1 \times \ldots \times 1 = 1.$$

Student Number Name

5. (5 marks) Suppose $X_1, \ldots, X_n$ are independent random variables, and $X_k \sim \text{Exponential}(k)$, i.e., X_k has pdf

$$f_k(x_k) = \begin{cases} ke^{-kx_k} & \text{for } x_k \geq 0 \\ 0 & \text{for } x_k < 0 \end{cases},$$

for $k = 1, \ldots, n$. Find $P(\min(X_1, \ldots, X_n) = X_n)$.

Solution: The event "$\min(X_1, \ldots, X_n) = X_n$" is the event $\{X_n < X_1, \ldots, X_n < X_{n-1}\}$. The n-dimensional integral giving the probability of this event can be written as

$$\begin{aligned}
P(X_n < X_1, \ldots, X_n < X_{n-1}) &= \int_0^\infty \int_{x_n}^\infty \ldots \int_{x_n}^\infty (e^{-x_1})(2e^{-2x_2}) \ldots (ne^{-nx_n}) dx_1 \ldots d_{x_{n-1}} dx_n \\
&= \int_0^\infty \int_{x_n}^\infty \ldots \int_{x_n}^\infty (2e^{-2x_2}) \ldots (ne^{-nx_n}) e^{-x_n} dx_2 \ldots d_{x_{n-1}} dx_n \\
&= \int_0^\infty \int_{x_n}^\infty \ldots \int_{x_n}^\infty (3e^{-3x_2}) \ldots (ne^{-nx_n}) e^{-x_n} e^{-2x_n} dx_3 \ldots d_{x_{n-1}} dx_n \\
&\vdots \\
&= \int_0^\infty ne^{-nx_n} e^{-x_n} e^{-2x_n} \ldots e^{-(n-1)x_n} dx_n \\
&= n \int_0^\infty e^{-(n(n+1)/2)x_n} dx_n = \frac{2n}{n(n+1)} = \frac{2}{n+1}.
\end{aligned}$$