Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Homework 3 Due February 14, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  (1) Use your own paper.
  (2) Use a tablet, such as an ipad.
  (3) Use document creation software, such as Word or LaTeX.
- Write your name and student number on the first page of each solution, and number your solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 30

Student Number Name

1. (6 marks) Let X_1, X_2, X_3 be independent, identically distributed continuous random variables. Find the probability that the second largest value (i.e., the median) is closer to the smallest value than to the largest value, when the common distribution of the X_i is

(a) (2 marks) the Uniform$(0, 1)$ distribution (a symmetry argument should suffice here);

(b) (4 marks) the Exponential(λ) distribution.

Student Number Name

2. (6 marks) Let $X_1, \ldots, X_n$ be a sequence of independent Uniform$(0,1)$ random variables, with $X_{(1)}, \ldots, X_{(n)}$ denoting their order statistics. Let A_n denote the expected area of the triangle formed by the vertices $(X_{(n-2)}, 0)$, $(X_{(n-1)}, X_{(n-1)})$, and $(X_{(n)}, 0)$. Find A_n (in terms of n) and show that $nA_n \to 1$ as $n \to \infty$.

Student Number Name

3. (6 marks) Let $X_1, \ldots, X_n$ be mutually independent Uniform(0,1) random variables. Find the probability that the interval $(\min(X_1, \ldots, X_n), \max(X_1, \ldots, X_n))$ contains the value 1/2 and find the smallest n such that this probability is at least 0.95.

Student Number Name

4. (6 marks) Let $X_1, \ldots, X_n$ be a sequence of independent Uniform$(0,1)$ random variables, with $X_{(1)}, \ldots, X_{(n)}$ denoting their order statistics. For fixed k let $g_n(x)$ denote the probability density function of $nX_{(k)}$. Find $g_n(x)$ and show that

$$\lim_{n\to\infty} g_n(x) = \begin{cases} \frac{x^{k-1}}{(k-1)!}e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

(note that as n increases the sample size also increases).

Student Number

Name

5. (6 marks) Let $X_1, X_2, \ldots$ be a sequence of independent random variables with the exponential distribution with mean 1, and let $X_{(n)} = \max(X_1, \ldots, X_n)$. For $x > 0$, show that

$$\lim_{n\to\infty} P(X_{(n)} - \ln n \le x) = \exp(-e^{-x}).$$