Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Homework 4   Due February 21, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  1. Use your own paper.
  2. Use a tablet, such as an ipad.
  3. Use document creation software, such as Word or LaTeX.
- Write your name and student number on the first page of each solution, and number your solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 28

Student Number Name

1. (6 marks) Let $X_1, \dots, X_n$ be independent exponential random variables with parameter λ, and let $X_{(1)}, \dots, X_{(n)}$ be their order statistics. Show that

$$Y_1 = nX_{(1)}, \quad Y_r = (n+1-r)(X_{(r)} - X_{(r-1)}), \quad r = 2, \dots, n$$

are also independent and have the same joint distribution as $X_1, \dots, X_n$. *Hint:* You may use the fact that the determinant of a lower triangular matrix (a square matrix whose entries above the main diagonal are all zero) is the product of the diagonal entries.

Student Number Name

2. (6 marks) Let X and Y be independent random variables, where X has a $N(0,1)$ distribution and Y has a χ^2 distribution with n degrees of freedom. Let $U = X/\sqrt{Y/n}$. Show that the pdf for U is

$$f_U(u) = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)\sqrt{\pi n}} \left(\frac{u^2}{n} + 1 \right)^{-(n+1)/2} \qquad \text{for } -\infty < u < \infty$$

(the distribution of U is known as the t distribution with n degrees of freedom). *Hint:* Define the auxilliary random variable $V = Y$. Find the joint pdf of U and V then use this to find the marginal pdf of U.

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3. (5 marks) For $n = 0, 1, 2, 3, \ldots$, show that

$$\Gamma(n + 1/2) = \frac{\sqrt{\pi}(2n)!}{4^n n!}.$$

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4. (6 marks) For $\alpha > 0$ and $\beta > 0$, show that $\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta)B(\alpha,\beta)$, where $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ ($B(\alpha,\beta)$ is called the *Beta Function*). *Hint:* Write

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty x^{\alpha-1}e^{-x}dx \int_0^\infty y^{\beta-1}e^{-y}dy = \int_0^\infty \int_0^\infty e^{-(x+y)}x^{\alpha-1}y^{\beta-1}dxdy$$

and change to the variables $u = x+y$, $v = x/(x+y)$.

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5. (5 marks) For 2 random variables X and Y with distribution functions F and G, respectively, we say that X is stochastically dominated by Y if $F(t) \geq G(t)$ for all $t \in \mathbb{R}$. Let $X_1, X_2, \ldots$ be a sequence of random variables such that X_n has a Gamma distribution with parameters n and λ, for some given $\lambda > 0$. Let F_n denote the distribution function of X_n. Compute $F_n(t) - F_{n+1}(t)$ for $t > 0$ and show that X_n is stochastically dominated by X_{n+1} for all n.