Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Homework 4 Solutions, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  (1) Use your own paper.
  (2) Use a tablet, such as an ipad.
  (3) Use document creation software, such as Word or LaTeX.
- Write your name and student number on the first page of each solution, and number your solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 28

Student Number Name

1. (6 marks) Let $X_1, \ldots, X_n$ be independent exponential random variables with parameter λ, and let $X_{(1)}, \ldots, X_{(n)}$ be their order statistics. Show that

$$Y_1 = nX_{(1)}, \quad Y_r = (n+1-r)(X_{(r)} - X_{(r-1)}), \quad r = 2, \ldots, n$$

are also independent and have the same joint distribution as $X_1, \ldots, X_n$. *Hint:* You may use the fact that the determinant of a lower triangular matrix (a square matrix whose entries above the main diagonal are all zero) is the product of the diagonal entries.

Solution: The transformation from the variables $(X_{(1)}, \ldots, X_{(n)})$ to $(Y_1, \ldots, Y_n)$ is a one-to-one transformation with inverse

$$\begin{aligned}
X_{(1)} &= \frac{1}{n}Y_1 \\
X_{(2)} &= \frac{1}{n}Y_1 + \frac{1}{n-1}Y_2 \\
X_{(3)} &= \frac{1}{n}Y_1 + \frac{1}{n-1}Y_2 + \frac{1}{n-2}Y_3 \\
&\vdots \\
X_{(r)} &= \frac{1}{n}Y_1 + \frac{1}{n-1}Y_2 + \ldots + \frac{1}{n+1-r}Y_r \\
&\vdots \\
X_{(n)} &= \frac{1}{n}Y_1 + \frac{1}{n-1}Y_2 + \ldots + \frac{1}{2}Y_{n-1} + Y_n.
\end{aligned}$$

The Jacobian of the transformation is

$$\mathbf{J} = \det \begin{bmatrix}
\frac{1}{n} & 0 & 0 & 0 & \ldots & 0 & 0 \\
\frac{1}{n} & \frac{1}{n-1} & 0 & 0 & \ldots & 0 & 0 \\
\frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & & 0 & 0 \\
\frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \ldots & & \frac{1}{2} & 0 \\
\frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \ldots & & \frac{1}{2} & 1
\end{bmatrix} = \frac{1}{n!}$$

The joint pdf of the order statistics $X_{(1)}, \ldots, X_{(n)}$ is

$$\begin{aligned}
f_{1 \ldots n}(x_1, \ldots, x_n) &= n! \prod_{i=1}^{n} f(x_i) \quad \text{for } x_1 < x_2 < \ldots < x_n \\
&= \begin{cases} n! \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} & \text{for } 0 < x_1 < x_2 < \ldots < x_n < \infty \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

where $f(\cdot)$ is the Exponential(λ) pdf of each X_i. Then by the multivariate change of variable formula the joint pdf of $Y_1, \ldots, Y_n$, say $g(y_1, \ldots, y_n)$, is given by

$$
\begin{aligned}
& g\,(\,y_1, \ldots, y_n) \\
&= f_{1\ldots n}\left(\frac{1}{n}y_1, \frac{1}{n}y_1 + \frac{1}{n-1}y_2, \ldots, \frac{1}{n}y_1 + \frac{1}{n-1}y_2 + \ldots + y_n\right)|\mathbf{J}| \\
&= \begin{cases} n!\lambda^n \exp\left\{-\lambda\left(\sum_{i=1}^n \frac{1}{n}y_1 + \sum_{i=2}^n \frac{1}{n-1}y_2 + \ldots + \sum_{i=n}^n y_n\right)\right\}\frac{1}{n!} & \text{for } y_1 > 0, \ldots, y_n > 0 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \lambda^n e^{-\lambda\sum_{i=1}^n y_i} & \text{for } y_1 > 0, \ldots, y_n > 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

But this is clearly the joint pdf of $X_1, \ldots, X_n$.

Student Number Name

2. (6 marks) Let X and Y be independent random variables, where X has a $N(0,1)$ distribution and Y has a χ^2 distribution with n degrees of freedom. Let $U = X/\sqrt{Y/n}$. Show that the pdf for U is

$$f_U(u) = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)\sqrt{\pi n}}\left(\frac{u^2}{n}+1\right)^{-(n+1)/2} \quad \text{for } -\infty < u < \infty$$

(the distribution of U is known as the t distribution with n degrees of freedom). *Hint:* Define the auxilliary random variable $V = Y$. Find the joint pdf of U and V then use this to find the marginal pdf of U.

Solution: Since X has a $N(0,1)$ distribution, Y has a χ^2 distribution with n degrees of freedom (i.e., a Gamma distribution with parameters $n/2$ and $1/2$), and X and Y are independent, the joint pdf of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\frac{1}{2^{n/2}\Gamma(n/2)}y^{(n/2)-1}e^{-y/2} & \text{for } -\infty < x < \infty \text{ and } 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Define $U = X/\sqrt{Y/n}$ and let $V = Y$. The inverse transformation is $X = U\sqrt{V/n}$ and $Y = V$, with Jacobian

$$\mathbf{J} = \det\begin{bmatrix} \sqrt{v/n} & u/2\sqrt{nv} \\ 0 & 1 \end{bmatrix} = \sqrt{v/n}.$$

The possible values of (U,V) are $-\infty < U < \infty$ and $V > 0$. From the (bivariate) change of variable formula, the joint density of U and V is given by

$$f_{UV}(u,v) = f_{XY}(u\sqrt{v/n}, v)|\mathbf{J}|.$$

For $-\infty < u < \infty$ and $v > 0$, this is

$$\begin{aligned} f_{UV}(u,v) &= \frac{1}{\sqrt{2\pi}}e^{-(u\sqrt{v/n})^2/2}\frac{1}{2^{n/2}\Gamma(n/2)}v^{(n/2)-1}e^{-v/2}\sqrt{\frac{v}{n}} \\ &= \frac{1}{\sqrt{2\pi}}e^{-u^2v/2n}\frac{1}{2^{n/2}\Gamma(n/2)}v^{(n/2)-1}e^{-v/2}\sqrt{\frac{v}{n}} \\ &= \frac{1}{\sqrt{2\pi n}}\frac{1}{2^{n/2}\Gamma(n/2)}v^{((n+1)/2)-1}e^{-v(u^2/2n+1/2)}. \end{aligned}$$

Then the marginal pdf of U is given by

$$\begin{aligned} f_U(u) &= \frac{1}{\sqrt{2\pi n}} \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty v^{((n+1)/2)-1} e^{-v(u^2/2n+1/2)} dv \\ &= \frac{1}{\sqrt{\pi n} 2^{(n+1)/2}\Gamma(n/2)} \frac{\Gamma((n+1)/2)}{(u^2/2n+1/2)^{(n+1)/2}} \\ &\quad \times \int_0^\infty \frac{(u^2/2n+1/2)^{(n+1)/2}}{\Gamma((n+1)/2)} v^{((n+1)/2)-1} e^{-v(u^2/2n+1/2)} dv, \end{aligned}$$

where the second equality is obtained by multiplying and dividing by $\frac{\Gamma((n+1)/2)}{(u^2/2n+1/2)^{(n+1)/2}}$. The integral is equal to one since the integrand is the pdf of a Gamma distribution with parameters $(n+1)/2$ and $u^2/2n+1/2$. Therefore, the pdf of U is given by

$$\begin{aligned} f_U(u) &= \frac{\Gamma((n+1)/2)}{\sqrt{\pi n} 2^{(n+1)/2}\Gamma(n/2)} \left(\frac{u^2}{2n} + \frac{1}{2} \right)^{-(n+1)/2} \\ &= \frac{\Gamma((n+1)/2)}{\sqrt{\pi n}\Gamma(n/2)} \left(\frac{u^2}{n} + 1 \right)^{-(n+1)/2}, \end{aligned}$$

as desired.

Student Number Name

3. (5 marks) For $n = 0, 1, 2, 3, \ldots$, show that

$$\Gamma(n + 1/2) = \frac{\sqrt{\pi}(2n)!}{4^n n!}.$$

Solution: We have seen in lecture that $\Gamma(1/2) = \sqrt{\pi}$. This handles the $n = 0$ case. For $n \geq 1$, using the recursive property of the Gamma function, $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ for $\alpha > 1$, we have

$$\begin{aligned}
\Gamma(n + 1/2) &= (n - 1 + 1/2)\Gamma(n - 1 + 1/2) \\
&\vdots \\
&= (n - 1 + 1/2)(n - 2 + 1/2)\ldots(n - n + 1/2)\Gamma(n - n + 1/2) \\
&= \left(\frac{2n - 1}{2}\right)\left(\frac{2n - 3}{2}\right)\ldots\left(\frac{2n - (2n - 1)}{2}\right)\sqrt{\pi} \\
&= \frac{(2n)!}{2^n(2n)(2n - 2)\ldots(2n - (2n - 2))}\sqrt{\pi} \\
&= \frac{(2n)!}{2^n 2^n(n)(n - 1)\ldots(n - (n - 1))}\sqrt{\pi} \\
&= \frac{(2n)!\sqrt{\pi}}{4^n n!}
\end{aligned}$$

Student Number Name

4. (6 marks) For $\alpha > 0$ and $\beta > 0$, show that $\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta)B(\alpha,\beta)$, where $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ ($B(\alpha,\beta)$ is called the *Beta Function*). *Hint:* Write

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty x^{\alpha-1}e^{-x}dx \int_0^\infty y^{\beta-1}e^{-y}dy = \int_0^\infty \int_0^\infty e^{-(x+y)}x^{\alpha-1}y^{\beta-1}dxdy$$

and change to the variables $u = x+y$, $v = x/(x+y)$.

Solution: Following the hint, we have

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty x^{\alpha-1}e^{-x}dx \int_0^\infty y^{\beta-1}e^{-y}dy = \int_0^\infty \int_0^\infty e^{-(x+y)}x^{\alpha-1}y^{\beta-1}dxdy$$

Making the substitution $u = x+y$ and $v = x/(x+y)$, the inverse transformation is $x = uv$ and $y = u(1-v)$, with Jacobian

$$\mathbf{J} = \det \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix} = -uv - u(1-v) = -u.$$

The limits for u and v are $0 < u < \infty$ and $0 < v < 1$. Therefore, we have

$$\begin{aligned} \Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty \int_0^\infty e^{-(x+y)}x^{\alpha-1}y^{\beta-1}dxdy \\ &= \int_0^1 \int_0^\infty e^{-(uv+u(1-v))}(uv)^{\alpha-1}(u(1-v))^{\beta-1}ududv \\ &= \int_0^1 \int_0^\infty v^{\alpha-1}(1-v)^{\beta-1}u^{\alpha+\beta-1}e^{-u}dudv \\ &= \left[\int_0^1 v^{\alpha-1}(1-v)^{\beta-1}dv\right]\left[\int_0^\infty u^{\alpha+\beta-1}e^{-u}du\right] \\ &= B(\alpha,\beta)\Gamma(\alpha+\beta), \end{aligned}$$

which gives the desired relation

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Student Number Name

5. (5 marks) For 2 random variables X and Y with distribution functions F and G, respectively, we say that X is stochastically dominated by Y if $F(t) \geq G(t)$ for all $t \in \mathbb{R}$. Let $X_1, X_2, \dots$ be a sequence of random variables such that X_n has a Gamma distribution with parameters n and λ, for some given $\lambda > 0$. Let F_n denote the distribution function of X_n. Compute $F_n(t) - F_{n+1}(t)$ for $t > 0$ and show that X_n is stochastically dominated by X_{n+1} for all n.

Solution: We show that X_n is stochastically dominated by X_{n+1} by computing $F_n(t) - F_{n+1}(t)$ and showing that it is nonnegative. For $t > 0$, we have

$$F_n(t) - F_{n+1}(t) = \int_0^t \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x} dx - \int_0^t \frac{\lambda^{n+1}}{n!} x^n e^{-\lambda x} dx$$

Doing integration by parts on the second integral (with $u = x^n$, $du = nx^{n-1}dx$, $dv = e^{-\lambda x}dx$ and $v = -\frac{1}{\lambda}e^{-\lambda x}$) gives

$$\begin{aligned}
F_n(t) - F_{n+1}(t) &= \int_0^t \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x} dx - \frac{\lambda^{n+1}}{n!} \left\{ -\frac{1}{\lambda} x^n e^{-\lambda x} \Big|_0^t + \frac{n}{\lambda} \int_0^t x^{n-1} e^{-\lambda x} dx \right\} \\
&= \int_0^t \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x} dx + \frac{(\lambda t)^n}{n!} e^{-\lambda t} - \int_0^t \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x} dx \\
&= \frac{(\lambda t)^n}{n!} e^{-\lambda t},
\end{aligned}$$

which is clearly positive for all $t > 0$. For $t \leq 0$ both $F_n(t)$ and $F_{n+1}(t)$ are equal to 0 and so their difference is 0. Therefore, we have that $F_n(t) - F_{n+1}(t) \geq 0$ for all $t \in \mathbb{R}$, so X_n is stochastically dominated by X_{n+1}.